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A_k singlarities of wave fronts

BY KENTARO SAJI†
Department of Mathematics, Faculty of Educaton
Gifu University, Yanagido 1-1, *Gifu* 151-1193, *Japan.*
e-mail: ksaji@gifu-u.ac.jp

MASAAKI UMEHARA‡
Department of Mathematics, Graduate School of Science
Osaka University, Toyonaka, Osaka 560-0043, *Japan.*
e-mail: umehara@math.sci.osaka-u.ac.jp

AND KOTARO YAMADA§
Faculty of Mathematics, Kyushu University
Fukuoka 812-8581, *Japan.*
e-mail: kotaro@math.kyushu-u.ac.jp

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Abstract

In this paper, we discuss the recognition problem for A_k-type singularities on wave fronts. We give computable and simple criteria of these singularities, which will play a fundamental role in generalizing the authors' previous work "the geometry of fronts" for surfaces. The crucial point to prove our criteria for A_k-singularities is to introduce a suitable parametrization of the singularities called the "kth KRSUY-coordinates" (see Section 3). Using them, we can directly construct a versal unfolding for a given singularity. As an application, we prove that a given nondegenerate singular point p on a real (resp. complex) hypersurface (as a wave front) in $\boldsymbol{R}^{n+1}$ (resp. $\boldsymbol{C}^{n+1}$) is differentiably (resp. holomorphically) right-left equivalent to the A_{k+1}-type singular point if and only if the linear projection of the singular set around p into a generic hyperplane $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$) is right-left equivalent to the A_k-type singular point in $\boldsymbol{R}^n$ (resp. $\boldsymbol{C}^n$). Moreover, we show that the restriction of a C^∞-map $f\colon \boldsymbol{R}^n \to \boldsymbol{R}^n$ to its Morin singular set gives a wave front consisting of only A_k-type singularities. Furthermore, we shall give a relationship between the normal curvature map and the zig-zag numbers (the Maslov indices) of wave fronts.

Dedicated to Professor Yoshiaki Maeda on the occasion of his sixtieth birthday.

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1. *Introduction*

Throughout this paper, we denote by $\boldsymbol{K}$ the real number field $\boldsymbol{R}$ or the complex number field $\boldsymbol{C}$. Let m and n be positive integers. A map $f\colon \boldsymbol{K}^m \to \boldsymbol{K}^n$ is called $\boldsymbol{K}$*-differentiable* if it is a C^∞-map when $\boldsymbol{K}=\boldsymbol{R}$, and is a holomorphic map when $\boldsymbol{K}=\boldsymbol{C}$. Let $\langle\ ,\ \rangle_{\boldsymbol{K}}$ be the $\boldsymbol{K}$*-inner product* given by

$$\langle X, Y\rangle_{\boldsymbol{K}} = \sum_{j=1}^{n} x_j y_j \qquad (X=(x_1,\dots,x_n),\ Y=(y_1,\dots,y_n)\in \boldsymbol{K}^n).$$

Let $P^n(\boldsymbol{K})$ be the $\boldsymbol{K}$-projective space and $\boldsymbol{K}^{n+1}\setminus\{\boldsymbol{0}\}\ni X\mapsto [X]\in P^n(\boldsymbol{K})$ the canonical projection. By using the above $\boldsymbol{K}$-inner product, the $\boldsymbol{K}$-projective cotangent bundle $PT^*\boldsymbol{K}^{n+1}$ has the following identification

$$PT^*\boldsymbol{K}^{n+1} = \boldsymbol{K}^{n+1}\times P^n(\boldsymbol{K}),$$

which has the canonical $\boldsymbol{K}$-contact structure. Let $U\subset \boldsymbol{K}^n$ be a domain and

$$L := (f,[\nu]) : U \longrightarrow \boldsymbol{K}^{n+1}\times P^n(\boldsymbol{K})$$

a Legendrian immersion, where ν is a locally defined $\boldsymbol{K}$-differentiable map into $\boldsymbol{K}^{n+1}\setminus\{\boldsymbol{0}\}$ such that

$$\langle df(\boldsymbol{u}), \nu\rangle_{\boldsymbol{K}} = 0 \qquad (\boldsymbol{u}\in TU).$$

In this situation, f is called a *wave front* or *front*, and ν is called the $\boldsymbol{K}$*-normal vector field* of f. If $\boldsymbol{K}=\boldsymbol{C}$, $\langle \nu,\nu\rangle_{\boldsymbol{K}}$ might vanish. A point $p\in U$ is called a *singular point* if the front f is not an immersion at p.

In this paper, we shall discuss the recognition problem for A_{k+1}-type singularities on wave fronts. These are fundamental singularities on wave fronts (see [**2**]). We give a simple and computable necessary and sufficient condition that a given singular point p on a hypersurface (as a wave front) in $\boldsymbol{K}^{n+1}$ is $\boldsymbol{K}$-right-left equivalent to the A_{k+1}-type singular point ($k\leqslant n$; Theorem 2·4, Corollaries 2·5, 2·6 and 2·8 in Section 2), where two $\boldsymbol{K}$-differentiable map germs $f_j\colon (U_j,p_j)\to(\boldsymbol{K}^{n+1},p'_j)$ $(j=1,2)$ are $\boldsymbol{K}$*-right-left equivalent* if there exist $\boldsymbol{K}$-diffeomorphism germs $\psi\colon (U_1,p_1)\to(U_2,p_2)$ and $\Psi : (\boldsymbol{K}^{n+1},p'_1)\to(\boldsymbol{K}^{n+1},p'_2)$ such that $\Psi\circ f_1 = f_2\circ\psi$ holds. Here, the A_{k+1}-type singularity (or A_{k+1}-front singularity) is a map germ defined by

$$X \longmapsto \left((k+1)t^{k+2}+\sum_{j=2}^{k}(j-1)t^j x_j,\ -(k+2)t^{k+1}-\sum_{j=2}^{k} j t^{j-1}x_j,\ X_1\right) \tag{1·1}$$

at the origin, where $X=(t,x_2,\dots,x_n)$, $X_1=(x_2,\dots,x_n)$. The image of it coincides with the discriminant set $\{F=F_t=0\}\subset(\boldsymbol{K}^{n+1};u_0,\dots,u_n)$ of the versal unfolding

$$F(t,u_0,\dots,u_n) := t^{k+2}+u_k t^k+\cdots+u_1 t+u_0. \tag{1·2}$$

By definition, A_1-front singularities are regular points. A 3/2-cusp in a plane is an A_2-front singularity and a swallowtail in $\boldsymbol{R}^3$ is an A_3-front singularity.

When $n=2$, useful criteria for cuspidal edges and swallowtails are given in [**7**]. We shall give a generalization of the criteria here. The crucial point to prove our criteria for A_{k+1}-singularities is to introduce the "kth KRSUY-coordinates" as a generalization of the coordinates for cuspidal edges and swallowtails in [**7**]. Using them, we can directly construct a versal unfolding whose discriminant set coincides with the given singularity.

As an application, when $\boldsymbol{K} = \boldsymbol{R}$, we show that the restriction of a C^∞-map $f\colon \boldsymbol{R}^{n+1} \to \boldsymbol{R}^{n+1}$ into its Morin singular set (see the appendix) gives a wave front consisting of only A_{k+1}-type singularities ($k \leqslant n$). Moreover, in the final section, we shall give a relationship between the normal curvature map and the zig-zag numbers (the Maslov indices) of wave fronts.

2. *Criteria for* A_{k+1}*-front singularities*

Let $U \subset (\boldsymbol{K}^n; x_1 \ldots, x_n)$ be a domain and $f\colon U \to \boldsymbol{K}^{n+1}$ a front. Since we will work on a local theory, we may assume that the $\boldsymbol{K}$-normal vector field ν of f is defined on U. We define a $\boldsymbol{K}$-differentiable function λ on U as the determinant

$$\lambda := \det(f_{x_1}, \ldots, f_{x_n}, \nu), \tag{2·1}$$

where $f_{x_j} := \partial f/\partial x_j$ $(j = 1, 2, \ldots, n)$. A point $p \in U$ is 1-*singular* (or *singular*) if $\lambda(p) = 0$, that is, p is a singular point. A point p is 1-*nondegenerate* (or *non-degenerate*) if p is 1-singular and the exterior derivative $d\lambda$ does not vanish at p. The following assertion is obvious:

LEMMA 2·1. *The definition of* 1*-nondegeneracy is independent of the choice of local coordinate system* $(x_1, \ldots, x_n)$ *and the choice of a* $\boldsymbol{K}$*-normal vector field* ν*. Moreover, if* p *is* 1*-nondegenerate, the* $\boldsymbol{K}$*-linear map* $df_p\colon T_pU \to \boldsymbol{K}^{n+1}$ *has a kernel of* $\boldsymbol{K}$*-dimension exactly one.*

If p is 1-nondegenerate, the singular set of $f\,(=: f_0)$

$$S_1 := S(f) = \{q \in U\,;\ \lambda(q) = 0\}$$

is an embedded $\boldsymbol{K}$*-differentiable* hypersurface of U near p. We denote by TS_1 the $\boldsymbol{K}$-differentiable tangent bundle of $\boldsymbol{K}$-differentiable manifold S_1. Then

$$f_1 := f|_{S_1} : S_1 \longrightarrow \boldsymbol{K}^{n+1}$$

is a $\boldsymbol{K}$-differentiable map. Since $\dim_{\boldsymbol{K}} \ker(df_p) = 1$, we can take a sufficiently small neighbourhood V $(\subset U)$ of p and non-zero $\boldsymbol{K}$-differentiable vector field η on $S_1 \cap V$ which belongs to the kernel of df, that is $df_q(\eta_q) = 0$ for $q \in S_1 \cap V$. We call η a *null vector field* of f. Moreover, we can construct a $\boldsymbol{K}$-differentiable vector field $\tilde{\eta}$ on V (called an *extended null vector field*) whose restriction $\tilde{\eta}|_{S_1 \cap V}$ on the singular set $S_1 \cap V$ gives a null vector field.

Remark 2·2. Since $\eta \in \ker(df)$ on S_1, we can write the components of η explicitly using determinants of $(n-1)$-submatrices of the Jacobian matrix of f. Then this explicit expression of η gives a $\boldsymbol{K}$-differentiable vector field on a sufficiently small neighbourhood of a singular point. Thus, we get an explicit procedure to construct $\tilde{\eta}$. For example, let f be a front $f = (f^1, f^2, f^3, f^4)\colon \boldsymbol{K}^3 \to \boldsymbol{K}^4$ such that $f^1_{x_1} f^2_{x_2} - f^1_{x_2} f^2_{x_1} \neq 0$. Then

$$\tilde{\eta} = \left(\det\begin{pmatrix} f^1_{x_2} & f^1_{x_3} \\ f^2_{x_2} & f^2_{x_3}\end{pmatrix}, -\det\begin{pmatrix} f^1_{x_1} & f^1_{x_3} \\ f^2_{x_1} & f^2_{x_3}\end{pmatrix}, \det\begin{pmatrix} f^1_{x_1} & f^1_{x_2} \\ f^2_{x_1} & f^2_{x_2}\end{pmatrix}\right)$$

is an extended null vector field.

Let $\lambda'\colon U \to \boldsymbol{K}$ and $f', f''\colon U \to \boldsymbol{K}^{n+1}$ be the derivatives $d\lambda(\tilde{\eta})$, $df(\tilde{\eta})$ and $df'(\tilde{\eta})$, respectively. Then by definition, we have

$$S_1 := \{q \in U;\ f'(q) = \boldsymbol{0}\}, \tag{2·2}$$

where $\mathbf{0} = (0, \ldots, 0)$. The following assertion holds:

LEMMA 2·3. *The singular set* $S_2 := S(f_1)$ *of* $f_1 \colon S_1 \to \boldsymbol{K}^{n+1}$ *has the following expressions*:

$$S_2 = \{q \in S_1\,;\ \eta_q \in T_q S_1\} = \{q \in S_1\,;\ \lambda'(q) = 0\} = \{q \in U\,;\ \lambda(q) = \lambda'(q) = 0\}$$
$$= \{q \in S_1\,;\ f''(q) = \mathbf{0}\} = \{q \in U\,;\ f'(q) = f''(q) = \mathbf{0}\}.$$

Proof. A point $q \in S_1$ belongs to S_2 if and only if there exists $\xi \in T_q S_1 \setminus \{\mathbf{0}\}$ such that $df_1(\xi) = 0$. Since $\dim_{\boldsymbol{K}} \ker(df_q) = 1$, the vector ξ must be proportional to η_q. Thus we get $S_2 = \{q \in S_1\,;\ \eta_q \in T_q S_1\}$. Since S_1 is a level set of λ, $\eta_q \in T_q S_1$ if and only if $d\lambda(\eta_q) = \lambda'(q) = 0$, which proves the first equation. By (2·2) and by $\eta \in TS_1$ on S_2, the derivative $f'' = df'(\eta)$ of f' with respect to η vanishes on S_2, that is $S_2 \subset \{q \in S_1\,;\ f''(q) = \mathbf{0}\}$. Next we suppose $f''(q) = \mathbf{0}$ $(q \in S_1)$. Since $\lambda' = 0$ on S_2, by taking a new coordinate system $(z_1, \ldots, z_n)$ on U such that $\partial/\partial z_n = \tilde{\eta}$, we have

$$\lambda'(q) = \{\varphi \det(f_{z_1}, \ldots, f_{z_{n-1}}, f', \nu)\}'(q) = \varphi(q) \det(f_{z_1}, \ldots, f_{z_{n-1}}, f'', \nu)(q) = 0,$$

where $\varphi := \det(\partial z_i/\partial x_j)$ is the Jacobian. Consequently $q \in S_2$ holds, that is $S_2 = \{q \in S_1\,;\ f''(q) = \mathbf{0}\}$.

A 1-nondegenerate singular point $p \in S_1$ is called 2-*singular* if $p \in S_2$. Moreover, a 2-singular point $p \in S_2$ is 2-*nondegenerate* if $(d\lambda')_p \neq 0$ on $T_p S_1$. In this case, S_2 is an embedded hypersurface of S_1 around p if it is 2-nondegenerate.

Now, we define the j-singularity and the j-nondegeneracy inductively $(2 \leqslant j \leqslant n)$: Firstly, we give the following notations:

$$\lambda^{(0)} := \lambda, \qquad \lambda^{(1)} := \lambda', \qquad \lambda^{(l)} := d\lambda^{(l-1)}(\tilde{\eta}),$$
$$f^{(0)} := f, \qquad f^{(1)} := f', \qquad f^{(l)} := df^{(l-1)}(\tilde{\eta}),$$

where $l = 1, 2, 3, \ldots$. We fix a $(j-1)$-nondegenerate singular point p of f. Then the $(j-1)$-st singular set

$$S_{j-1} := S(f_{j-2})$$

is an embedded hypersurface of S_{j-2} around p. (Here we replace U by a sufficiently small neighbourhood of p at each induction step if necessary. In particular, we may assume $S_0 := U$.) We set

$$f_{j-1} := f|_{S_{j-1}} : S_{j-1} \longrightarrow \boldsymbol{K}^{n+1}.$$

As in the proof of Lemma 2·3, by an inductive use of the identity

$$\lambda^{(j-1)}(q) = \{\varphi \det(f_{z_1}, \ldots, f_{z_{n-1}}, f', \nu)\}^{(j-1)}(q)$$
$$= \varphi(q) \det(f_{z_1}, \ldots, f_{z_{n-1}}, f^{(j)}, \nu)(q),$$

$S_j := S(f_{j-1})$ satisfies

$$S_j = \{q \in S_{j-1}\,;\ \eta_q \in T_q S_{j-1}\}$$
$$= \{q \in S_{j-1}\,;\ \lambda^{(j-1)}(q) = 0\} = \{q \in U\,;\ \lambda(q) = \cdots = \lambda^{(j-1)}(q) = 0\}$$
$$= \{q \in S_{j-1}\,;\ f^{(j)}(q) = \mathbf{0}\} = \{q \in U\,;\ f'(q) = \cdots = f^{(j)}(q) = \mathbf{0}\}. \qquad (2·3)$$

Each point of S_j is called a *j-singular point*. A point $q \in S_j$ is called *j-nondegenerate* if $(d\lambda^{(j-1)})_q \neq 0$ on $T_q S_{j-1}$, that is

$$T_q S_{j-1} \not\subset \ker(d\lambda^{(j-1)})_q. \tag{2·4}$$

These definitions do not depend on the choice of a coordinate system $(x_1, \ldots, x_n)$ of U. Under these notations, a criterion for A_{k+1}-front singularities is stated as follows:

THEOREM 2·4. *Let $f:(U, p) \to (\boldsymbol{K}^{n+1}, f(p))$ be a front, where U is a domain in $\boldsymbol{K}^n$. Then f at p is $\boldsymbol{K}$-right-left equivalent to the A_{k+1}-front singularity if and only if p is k-nondegenerate but is not $(k+1)$-singular ($k \leqslant n$).*

Though k-singular points are defined only for $k \leqslant n$, we define any points are not $(n+1)$-singular. This theorem is proved in Sections 3 and 4.

COROLLARY 2·5. *Let $f: U \to \boldsymbol{K}^{n+1}$ be a front. Then f at $p \in U$ is $\boldsymbol{K}$-right-left equivalent to the A_{k+1}-front singularity ($1 \leqslant k \leqslant n$) if and only if*

$$\lambda = \lambda' = \cdots = \lambda^{(k-1)} = 0, \quad \lambda^{(k)} \neq 0 \tag{2·5}$$

at p and the Jacobian matrix of $\boldsymbol{K}$-differentiable map $\Lambda := (\lambda, \lambda', \ldots, \lambda^{(k-1)}) : U \to \boldsymbol{K}^k$ is of rank k at p.

COROLLARY 2·6. *Let U be a domain in $\boldsymbol{K}^n$, $f: U \to \boldsymbol{K}^{n+1}$ a front, and $p \in U$ a nondegenerate singular point. Take a local tangent frame field $\{v_1, \ldots, v_{n-1}\}$ of the singular set $S(f)$ (a smooth hypersurface in U) around p and set*

$$\mu := \det(v_1, \ldots, v_{n-1}, \eta), \tag{2·6}$$

as the determinant function on $\boldsymbol{K}^n$ where η is a null vector field. Then p is 2-singular if and only if $\mu(p) = 0$. In particular, p is $\boldsymbol{K}$-right-left equivalent to the A_2-singularity if and only if $\mu(p) \neq 0$. Moreover, a 2-singular point p is 2-nondegenerate if and only if $d\mu(T_p S(f)) \neq \{0\}$. Furthermore, p is $\boldsymbol{K}$-right-left equivalent to the A_3-singularity if and only if $\mu(p) = 0$ and $d\mu(\eta_p) \neq 0$ hold. (If $\mu(p) = 0$, then the null vector η_p at p is tangent to $S(f)$ and $d\mu(\eta_p)$ is well-defined.)

Remark 2·7. These assertions for $n = 2$ have been already proved in [**7**]. The criterion for A_2-singularities for general n has been shown in [**10**].

Let $f: U \to \boldsymbol{K}^{n+1}$ a front, and $p \in U$ a 2-nondegenerate singular point. Then $S_2 = S(f_1)$ is a submanifold of codimension 2 in U and Lemma 2·3 yields that the null vector field η is a tangent vector field of S_2, and we can set

$$\mu' = d\mu(\eta), \quad \mu'' = d\mu'(\eta), \ldots, \mu^{(k-1)} = d\mu^{(k-2)}(\eta).$$

Then these functions $\mu', \mu'', \ldots, \mu^{(k-1)}$ are C^∞-functions on S_2.

COROLLARY 2·8. *Let U be a domain in $\boldsymbol{K}^n$, $f: U \to \boldsymbol{K}^{n+1}$ a front, and $p \in U$ a 2-nondegenerate singular point. Then p is $\boldsymbol{K}$-right-left equivalent to the A_{k+1}-singularity ($3 \leqslant k \leqslant n$) if and only if*

$$\mu = \mu' = \cdots = \mu^{(k-2)} = 0, \qquad \mu^{(k-1)} \neq 0 \qquad \text{at } p \tag{2·7}$$

hold, and the Jacobian matrix of the map $\Phi_\mu := (\mu', \ldots, \mu^{(k-2)}): S_2 \to \boldsymbol{K}^{k-2}$ is of rank $k-2$ at p.

Remark 2·9. In Corollary 2·8, the criterion for A_4-singularities reduces to 2-nondegeneracy and

$$\mu(p) = \mu'(p) = 0, \qquad \mu''(p) \neq 0,$$

since the condition $\mu''(p) \neq 0$ implies that the map $\Phi_\mu = \mu'$ is of rank 1 at p.

Proofs of these assertions are given in Section 4.

Let $f: U \to \boldsymbol{K}^{n+1}$ be a front and p a 1-nondegenerate singular point. Then we can consider the restriction $f_1: S(f) \to \boldsymbol{K}^{n+1}$ of f into S_1. Let us denote the limiting tangent hyperplane by

$$\nu(p)^\perp := \left\{\boldsymbol{v} \in T_{f(p)}\boldsymbol{K}^{n+1}\,;\ \langle \boldsymbol{v}, \nu(p)\rangle_{\boldsymbol{K}} = 0\right\}.$$

The following assertion can be proved straightforwardly.

COROLLARY 2·10. *Let $f: U \to \boldsymbol{K}^{n+1}$ be a front and $p \in U$ a 1-nondegenerate singular point. For any vector $\boldsymbol{n} \notin \nu(p)^\perp$, the following are equivalent*:
(1) *f at p is an A_{k+1}-front singularity ($k \leqslant n$);*
(2) *the projection $\pi_{\boldsymbol{n}} \circ f|_{S(f)}$ at p is an A_k-front singularity ($k \leqslant n$).*
Here, $\pi_{\boldsymbol{n}}$ is the normal projection with respect to $\boldsymbol{n}$ to the hyperplane

$$\boldsymbol{n}^\perp := \left\{\boldsymbol{v} \in T_{f(p)}\boldsymbol{K}^{n+1}\,;\ \langle \boldsymbol{v}, \boldsymbol{n}\rangle_{\boldsymbol{K}} = 0\right\}$$

and an A_1-front singularity means a regular point.

The following assertion follows immediately from Theorem A1 in the Appendix.

COROLLARY 2·11. *Let Ω be a domain in $\boldsymbol{K}^{n+1}$ and $f: \Omega \to \boldsymbol{K}^{n+1}$ a $\boldsymbol{K}$-differentiable map. Suppose that $p \in \Omega$ is a 1-nondegenerate singular point, namely, the exterior derivative of the Jacobian of f does not vanish at p. Then the following are equivalent*:
(1) *p is an A_k-Morin singular point of f;*
(2) *$f|_{S(f)}$ is a front, and p is an A_k-front singularity of $f|_{S(f)}$.*

Here, the A_k-Morin singularities are defined in the appendix, and the A_0-Morin singularity means a regular point.

3. *Adopted coordinates and k-nondegeneracy*

In [**7**], a certain kind of special coordinate system was introduced to treat the recognition problem of cuspidal edges and swallowtails in $\boldsymbol{R}^3$. In this section, we give a generalization of such a coordinate system around a singular point, which will play a crucial role.

LEMMA 3·1 (Existence of the kth KRSUY-coordinate system). *Let $f: U \to \boldsymbol{K}^{n+1}$ be a front and $p \in U$ a k-nondegenerate singular point ($1 \leqslant k \leqslant n$). Then there exists a $\boldsymbol{K}$-differentiable coordinate system $(z_1, \ldots, z_n)$ around p and a (non-degenerate) $\boldsymbol{K}$-affine transformation $\Theta : \boldsymbol{K}^{n+1} \to \boldsymbol{K}^{n+1}$ such that*

$$\hat{f}(z_1, \ldots, z_n) = (\hat{f}^1, \ldots, \hat{f}^{n+1}) = \Theta \circ f(z_1, \ldots, z_n)$$

satisfies the following properties:
(P_0) *the point p corresponds to $(z_1, \ldots, z_n) = (0, \ldots, 0)$;*
(P_1) *$\partial_{z_n} := \partial/\partial z_n$ gives an extended null vector field on U;*
(P_2) *if $k \geqslant 2$, then the $\boldsymbol{K}$-vector space T_pS_i ($i = 1, \ldots, k-1$) is spanned by $\partial_{z_{i+1}}, \ldots, \partial_{z_n}$, that is*

$$T_pS_i = \mathrm{Span}_{\boldsymbol{K}}\left\{\partial_{z_{i+1}}, \ldots, \partial_{z_n}\right\},$$

where $\partial_{z_j} := \partial/\partial z_j$, $j = 1, \ldots, n$;

(P_3) *suppose* $k < n$. *If* p *is* $(k+1)$*-singular, then* $T_pS_k = \mathrm{Span}_{\boldsymbol{K}}\{\partial_{z_{k+1}}, \dots, \partial_{z_n}\}$. *On the other hand, if* p *is not* $(k+1)$*-singular,* $T_pS_k = \mathrm{Span}_{\boldsymbol{K}}\{\partial_{z_k}, \dots, \partial_{z_{n-1}}\}$*;*

(P_4) $\hat{f}(p) = \mathbf{0}$ *and* $\nu(p) = (1, 0, \dots, 0)$*;*

(P_5) $\hat{f}(z_1, \dots, z_n) = (\hat{f}^1(z_1, \dots, z_n), \hat{f}^2(z_1, \dots, z_n), z_1, \dots, z_{n-1})$ *for* $n \geqslant 2$ *and* $\hat{f}(z_1) = (\hat{f}^1(z_1), \hat{f}^2(z_1))$ *for* $n = 1$*;*

(P_6) *if* $n \geqslant 2$, $(\hat{f}^1)_{z_i}(p) = (\hat{f}^2)_{z_i}(p) = 0$ *holds for* $i = 1, \dots, n-1$.

We call a coordinate system on U satisfying the properties (P_0), (P_1), (P_2) and (P_3) a "*k-adopted coordinate system*", and we call by the "*kth KRSUY-coordinate system*" a pair of coordinate systems on U and $\boldsymbol{K}^{n+1}$ satisfying all above conditions. (In [**7**], the existence of the coordinates for $k = 1, 2$ and $n = 2$ was shown.)

Proof of Lemma 3·1. By replacing U to be a smaller neighbourhood if necessary, we can take an extended null vector field $\tilde{\eta}$ on U (see Remark 2·2). Now we assume that $p \in S_k$ is k-nondegenerate. Then S_k is a submanifold of U of codimension k. So we can take a basis $\{\xi_1, \dots, \xi_{n-1}, \tilde{\eta}_p\}$ of T_pU such that

$$\begin{cases} \mathrm{Span}_{\boldsymbol{K}}\{\xi_k, \dots, \xi_{n-1}\} = T_pS_k & \text{if } k < n \text{ and } p \text{ is not } k\text{-singular}, \\ \mathrm{Span}_{\boldsymbol{K}}\{\xi_{k+1}, \dots, \xi_{n-1}, \tilde{\eta}_p\} = T_pS_k & \text{if } k < n \text{ and } p \text{ is } k\text{-singular}, \end{cases}$$

and

$$\mathrm{Span}_{\boldsymbol{K}}\{\xi_{i+1}, \dots, \xi_{n-1}, \tilde{\eta}_p\} = T_pS_i \qquad (1 \leqslant i \leqslant k-1 \quad \text{and} \quad k \geqslant 2).$$

We now take a local coordinate system $(s_1, s_2, \dots, s_n)$ around p such that $\partial_{s_i}(p) = \xi_i$ $(i = 1, \dots, n-1)$, and consider the following two involutive distributions T_1 and T_2 of rank $n-1$ and 1 respectively

$$T_1(x) := \mathrm{Span}_{\boldsymbol{K}}\big\{\partial_{s_1}(x), \partial_{s_2}(x), \dots, \partial_{s_{n-1}}(x)\big\}, \quad T_2(x) := \mathrm{Span}_{\boldsymbol{K}}\{\tilde{\eta}_x\} \quad (x \in V),$$

where V is a sufficiently small neighbourhood of p. Since two distributions $T_1(p)$ and $T_2(p)$ span T_pV, the lemma in [**6**, page 182] yields the existence of local coordinate system $(w_1, w_2, \dots, w_n)$ around p such that

$$T_1 := \mathrm{Span}_{\boldsymbol{K}}\big\{\partial_{w_1}, \partial_{w_2}, \dots, \partial_{w_{n-1}}\big\}, \qquad \partial_{w_n} \in \mathrm{Span}_{\boldsymbol{K}}\{\tilde{\eta}\}\ (= T_2).$$

Moreover, by a suitable affine transformation of $\boldsymbol{K}^n$, we may assume that $w_i(p) = 0$ $(i = 1, \dots, n)$ and $\partial_{w_i}(p) = \xi_i$ $(i = 1, \dots, n-1)$. Replacing U to be sufficiently smaller, we may assume that this new coordinate system $(w_1, \dots, w_n)$ is defined on U. Moreover, we may reset $\tilde{\eta} = \partial_{w_n}$. Thus $(U; w_1, \dots, w_n)$ satisfies (P_0), (P_1), (P_2) and (P_3). By a suitable affine transformation of $\boldsymbol{K}^{n+1}$, we may assume that

$$f(p) = \mathbf{0}, \quad \nu(p) = \boldsymbol{e}_1, \qquad \frac{\partial f}{\partial w_i}(p) = \boldsymbol{e}_{i+2} \quad (i = 1, \dots, n-1), \tag{3·1}$$

where $\boldsymbol{e}_j$ denotes the jth canonical vector

$$\boldsymbol{e}_j := (0, \dots, 0, 1, 0, \dots, 0) \qquad (j = 1, 2, \dots, n+1) \tag{3·2}$$

of $\boldsymbol{K}^{n+1}$. We set $f = (f^1, \dots, f^{n+1})$. Then it holds that

$$\frac{\partial f^{j+2}}{\partial w_l}(p) = \delta^j_l \qquad (j, l = 1, \dots, n-1), \tag{3·3}$$

where δ_l^j is Kronecker's delta. Define new functions by

$$g_j(z_1, \dots, z_{n-1}; w_1, \dots, w_n) := f^{j+2}(w_1, \dots, w_n) - z_j \qquad (j = 1, \dots, n-1).$$

Since $\partial g_j/\partial w_l(0) = \delta_l^j$ for $(j, l = 1, \dots, n-1)$, the implicit function theorem yields that there exist functions $\varphi_j(z_1, \dots, z_{n-1}, w_n)$ $(j = 1, \dots, n-1)$ such that

$$g_j(z_1, \dots, z_{n-1}, \varphi_1(z_1, \dots, z_{n-1}, w_n), \dots, \varphi_{n-1}(z_1, \dots, z_{n-1}, w_n), w_n) = 0.$$

If we set $w_n := z_n$ and

$$w_j := \varphi_j(z_1, \dots, z_{n-1}, w_n) \qquad (j = 1, \dots, n-1),$$

then $(z_1, \dots, z_n)$ gives a new local coordinate system of U around p such that

$$z_j = f^{j+2}\big(\varphi_1(z_1, \dots, z_{n-1}, z_n), \dots, \varphi_{n-1}(z_1, \dots, z_{n-1}, z_n), z_n\big), \tag{3·4}$$

$$\frac{\partial w_j}{\partial z_l}(p) = \delta_l^j \qquad (j = 1, \dots, n-1,\ l = 1, \dots, n). \tag{3·5}$$

Since ∂_{w_n} is the null direction, we have $\partial f/\partial w_n = 0$. Differentiating (3·4) with respect to z_n, we have

$$0 = \sum_{l=1}^{n-1} \frac{\partial f^{j+2}}{\partial w_l}\frac{\partial \varphi_l}{\partial z_n} + \frac{\partial f^{j+2}}{\partial w_n} = \sum_{l=1}^{n-1} \frac{\partial f^{j+2}}{\partial w_l}\frac{\partial \varphi_l}{\partial z_n} \qquad \text{on } S_1.$$

Since the matrix $(\partial f^{j+2}/\partial w_l)_{j,l=1,\dots,n-1}$ is regular near p by (3·3),

$$\frac{\partial w_l}{\partial z_n} = \frac{\partial \varphi_l}{\partial z_n} = 0 \qquad (\text{on } S_1 \text{ near } p)$$

hold for $l = 1, \dots, n-1$. In particular, (3·5) holds for $j = n$.

Thus if we set

$$\hat{f}(z_1, \dots, z_n) := f\big(\varphi_1(z_1, \dots, z_n), \dots \varphi_{n-1}(z_1, \dots, z_n), z_n\big),$$

then

$$\frac{\partial \hat{f}}{\partial z_n} = \sum_{j=1}^{n-1} \frac{\partial f}{\partial w_j}\frac{\partial \varphi_j}{\partial z_n} + \frac{\partial f}{\partial w_n} = \mathbf{0} \qquad \text{on } S_1 \text{ near } p,$$

holds which implies (P$_1$). By (3·5), $\partial_{z_j}(p) = \partial_{w_j}(p)$ for $j = 1, \dots, n$. Hence we have (P$_2$) and (P$_3$). Moreover,

$$\frac{\partial \hat{f}}{\partial z_i}(p) = \left(\sum_{j=1}^{n-1} \frac{\partial f}{\partial w_j}(p)\frac{\partial w_j}{\partial z_i}(p)\right) + \frac{\partial f}{\partial w_n}(p) = \frac{\partial f}{\partial w_i}(p) = \boldsymbol{e}_{i+2},$$

which implies (P$_6$). Now (P$_4$) and (P$_5$) follow immediately from (3·1) and (3·4) respectively.

Now we use the notation $' = \partial/\partial z_n$ under the k-adopted coordinate system.

LEMMA 3·2. *Let p be a k-nondegenerate singular point of f. Under a kth adopted coordinate system $(z_1, \dots, z_n)$, the following hold*:

(1) *if $k = 1$, then p is 2-singular if and only if $\lambda'(p) = 0$. Moreover $p (\in S_2)$ is 2-nondegenerate if and only if*

$$(\lambda'_{z_2}(p), \dots, \lambda'_{z_{n-1}}(p), \lambda''(p)) \neq \mathbf{0};$$

(2) *if* $k \geqslant 2$, *then* $\lambda^{(m-2)}_{z_{m-1}}(p) \neq 0$ *and*

$$\lambda^{(m-2)}_{z_m}(p) = \cdots = \lambda^{(m-2)}_{z_{n-1}}(p) = \lambda^{(m-1)}(p) = 0 \tag{3·6}$$

hold for $m = 2, \ldots, k$. *Moreover* p *is* $(k+1)$*-singular* $(k < n)$ *if and only if* $\lambda^{(k)}(p) = 0$. *Furthermore* $p(\in S_{k+1})$ *is* $(k+1)$*-nondegenerate if and only if*

$$\left(\lambda^{(k)}_{z_{k+1}}(p), \ldots, \lambda^{(k)}_{z_{n-1}}(p), \lambda^{(k+1)}(p)\right) \neq \mathbf{0}.$$

Proof. When $1 \leqslant k < n$, p is $(k+1)$-singular if and only if $\lambda^{(k)}(p) = 0$ by (2·3). Suppose now p is $(k+1)$-singular $(k < n)$. By definition, p is $(k+1)$-nondegenerate if and only if $T_pS_k \not\subset \ker(d\lambda^{(k)})_p$. Since $\mathrm{Span}_{\boldsymbol{K}}\{\partial_{z_{k+1}}, \ldots, \partial_{z_n}\} = T_pS_k$ by (P_2), this is equivalent to $\left(\lambda^{(k)}_{z_{k+1}}, \ldots, \lambda^{(k)}_{z_{n-1}}, \lambda^{(k+1)}\right)(p) \neq \mathbf{0}$, which proves (1) and the latter part of (2). Next, we assume $2 \leqslant m \leqslant k \leqslant n$ and prove the first part of (2): by (2·3), we have $S_{m-1} = \{q \in S_{m-2}\,;\ \lambda^{(m-2)}(q) = 0\}$. By the assumption (P_2), we have $T_pS_{m-1} = \mathrm{Span}_{\boldsymbol{K}}\{\partial_{z_m}, \ldots, \partial_{z_n}\}$. In particular, $\lambda^{(m-2)}$ is constant along these directions, which implies (3·6). On the other hand, since $k \geqslant m \geqslant 2$, p is $(m-1)$-nondegenerate. By (2·4), we have $T_pS_{m-2} \not\subset \ker(d\lambda^{(m-2)})_p$, which implies $\lambda^{(m-2)}_{z_{m-1}}(p) \neq 0$ because of (3·6).

LEMMA 3·3. *Suppose that* $p \in U$ *is a singular point of* f *which is* k*-nondegenerate but not* $(k+1)$*-singular. Then under the* k*th adopted coordinate system,* $f^{(k+1)}(p) \neq \mathbf{0}$ *holds. Moreover, if* $k \geqslant 2$, *then* $f^{(m-1)}_{z_{m-1}}(p) \neq \mathbf{0}$ *and*

$$f^{(m-1)}_{z_m}(p) = \cdots = f^{(m-1)}_{z_{n-1}}(p) = f^{(m)}(p) = \mathbf{0} \tag{3·7}$$

hold for $m = 2, \ldots, k$.

Proof. Since $p \in S_k \setminus S_{k+1}$, (2·3) yields that

$$f'(p) = \cdots = f^{(k)}(p) = \mathbf{0}, \quad f^{(k+1)}(p) \neq \mathbf{0}.$$

Suppose that $k \geqslant m \geqslant 2$. Then p is m-nondegenerate. By (2·3), the map $f^{(m-1)}$ vanishes along S_{m-1}. Since T_pS_{m-1} is spanned by $\partial_{z_m}, \ldots, \partial_{z_n}$ because of (P_2), we have (3·7). On the other hand, since $f' = \cdots = f^{(m-1)} = \mathbf{0}$ at p, Lemma 3·2 yields that

$$0 \neq \lambda^{(m-2)}_{z_{m-1}}(p) = \det\left(f_{z_1}, \ldots, f_{z_{n-1}}, f^{(m-1)}_{z_{m-1}}, \nu\right)(p),$$

which implies that $f^{(m-1)}_{z_{m-1}}(p) \neq \mathbf{0}$.

4. *Proof of the criteria*

Let $f\colon U \to \boldsymbol{K}^{n+1}$ be a front and ν be a $\boldsymbol{K}$-normal vector. Let $p \in U$ be a singular point of f which is k-nondegenerate but not $(k+1)$-singular. We denote the k-th KRSUY coordinates by $(U; z_1, \ldots, z_n)$ and $(\boldsymbol{K}^{n+1}; x_1, \ldots, x_{n+1})$. Set

$$X := (x_1, x_2), \quad Y := (z_1, \ldots, z_{n-1}), \quad (X, Y) = (x_1, x_2, z_1, \ldots, z_{n-1}).$$

We define a function $\Phi\colon \boldsymbol{K}^{n+1} \times \boldsymbol{K} \to \boldsymbol{K}$ by

$$\Phi(X, Y, z_n) := \langle \nu(Y, z_n), f(Y, z_n) - (X, Y)\rangle_{\boldsymbol{K}}. \tag{4·1}$$

Then we have the following:

PROPOSITION 4·1. *The discriminant set*

$$D_\Phi := \left\{(X, Y) \in \boldsymbol{K}^{n+1}\,;\ \Phi(X, Y, t) = \frac{\partial \Phi}{\partial t}(X, Y, t) = 0 \text{ for some } t\right\}$$

coincides with the image $\mathrm{Im}(f)$ *of* f.

Proof. One can easily prove that $D_\Phi \supset \mathrm{Im}(f)$. Next we show $D_\Phi \subset \mathrm{Im}(f)$. Suppose $(X, Y) \in D_\Phi$. Since $\langle f_{z_n}, \nu\rangle_{\boldsymbol{K}} = 0$, $(X, Y) \in D_\Phi$ is equivalent to

$$\langle \nu, f - (X, Y)\rangle_{\boldsymbol{K}} = 0 \quad \text{and} \quad \langle \nu', f - (X, Y)\rangle_{\boldsymbol{K}} = 0. \tag{4·2}$$

Since we are using the kth KRSUY-coordinates, (4·2) reduces to

$$\langle \nu, (f^1 - x_1, f^2 - x_2, 0, \dots, 0)\rangle_{\boldsymbol{K}} = \langle \nu', (f^1 - x_1, f^2 - x_2, 0, \dots, 0)\rangle_{\boldsymbol{K}} = 0. \tag{4·3}$$

By the following Lemma 4·2, the first two components of ν, ν' are linearly independent near p as vectors in $(\boldsymbol{K}^2; x_1, x_2)$. Hence by (4·3), we have $x_1 = f^1$, $x_2 = f^2$. Since

$$(X, Y) = (f^1(Y, z_n), f^2(Y, z_n), Y) = f(Y, z_n) \in \mathrm{Im}(f)$$

holds, we have $D_\Phi \subset \mathrm{Im}(f)$.

LEMMA 4·2. *The vectors $f_{z_1}, \dots, f_{z_{n-1}}, \nu, \nu'$ are linearly independent near p. Moreover, if we write $\nu = (\nu^1, \dots, \nu^{n+1})$, then (ν^1, ν^2) and $((\nu^1)', (\nu^2)')$ are linearly independent in $\boldsymbol{K}^2$ near p.*

Proof. Clearly, $f_{z_1}, \dots, f_{z_{n-1}}, \nu$ are linearly independent. Since f is a front and $f'(p) = \mathbf{0}$, we have $\nu' \notin \mathrm{Span}_{\boldsymbol{K}}\{\nu\}$. We assume $\nu' \in \mathrm{Span}_{\boldsymbol{K}}\{f_{z_1}, \dots, f_{z_{n-1}}, \nu\}$ at p. Since $\langle \nu', f_{z_i}\rangle_{\boldsymbol{K}} = -\langle \nu, f'_{z_i}\rangle_{\boldsymbol{K}} = \langle \nu_{z_i}, f'\rangle_{\boldsymbol{K}} = 0$, $\nu' \in \mathrm{Span}_{\boldsymbol{K}}\{\nu\}$, a contradiction. Moreover, since $f_{z_j}(p) = \boldsymbol{e}_{j+2}$ by (P_5), (ν^1, ν^2) and $\big((\nu^1)', (\nu^2)'\big)$ are linearly independent. Then the second part of the lemma follows.

LEMMA 4·3. *Suppose that $p \in S(f)$ is k-nondegenerate but not $(k+1)$-singular. Under the kth KRSUY-coordinates at p, it holds that:*
(1) $\Phi^{(m)}(\mathbf{0}, 0) = 0$ $(m = 0, \dots, k+1)$ *and* $\Phi^{(k+2)}(\mathbf{0}, 0) \neq 0$, *where* $\mathbf{0}$ *is the origin of* $\boldsymbol{K}^{n+1}$;
(2) *the Jacobian matrix of the map* $(\Phi, \Phi', \dots, \Phi^{(k)})\colon \boldsymbol{K}^{n+2} \to \boldsymbol{K}^{k+1}$ *is of rank* $k+1$ *at the origin.*

Proof. By identifying $\bigwedge^n \boldsymbol{K}^{n+1}$ with $\boldsymbol{K}^{n+1}$ with respect to $\langle\ ,\ \rangle_{\boldsymbol{K}}$, $f_{z_1}, \dots, f_{z_{n-1}}, \nu$ and $f_{z_1} \wedge \cdots \wedge f_{z_{n-1}} \wedge \nu$ are linearly independent near p, where $\wedge$ denotes the exterior product. Then, one can write

$$\nu' = -a f_{z_1} \wedge \cdots \wedge f_{z_{n-1}} \wedge \nu + \left(\sum_{l=1}^{n-1} b_l f_{z_l}\right) + c\nu, \tag{4·4}$$

for some functions $a, b_1, \dots, b_{n-1}, c$ in $(z_1, \dots, z_n)$. By Lemma 4·2, we have

$$a \neq 0 \qquad \text{near } p. \tag{4·5}$$

Here, on the singular set S_1, we have

$$0 = \langle \nu_{z_j}, f'\rangle_{\boldsymbol{K}} = \langle \nu', f_{z_j}\rangle_{\boldsymbol{K}} = \sum_{l=1}^{n-1} b_l \langle f_{z_l}, f_{z_j}\rangle_{\boldsymbol{K}} \qquad (j = 1, \dots, n-1).$$

Hence we have

$$b_j = 0 \quad (j = 1, \dots, n-1) \qquad \text{on } S_1 \text{ near } p. \tag{4·6}$$

By the definition of λ, we can write

$$\langle \nu', f'\rangle_{\boldsymbol{K}} = a\lambda + \sum_{l=1}^{n-1} b_l \langle f_{z_l}, f'\rangle_{\boldsymbol{K}}. \tag{4·7}$$

Since $\langle \nu', f' \rangle_K = 0$, differentiating (4·1) with respect to z_n, we have

$$\begin{aligned}\Phi' &= \langle \nu', f - (X, Y) \rangle_K + \langle \nu, f' \rangle_K = \langle \nu', f - (X, Y) \rangle_K,\\ \Phi'' &= \langle \nu'', f - (X, Y) \rangle_K + 2\langle \nu', f' \rangle_K + \langle \nu, f'' \rangle_K \\ &= \langle \nu'', f - (X, Y) \rangle_K + \langle \nu', f' \rangle_K.\end{aligned} \tag{4·8}$$

Then by Lemmas 3·2, 3·3 and (4·7), we have

$$\Phi^{(m+2)} = (\langle \nu', f' \rangle_K)^{(m)} = \left(a\lambda + \sum_{l=1}^{n-1} b_l \langle f_{z_l}, f' \rangle_K\right)^{(m)} = a\lambda^{(m)} \tag{4·9}$$

at $(\mathbf{0}, 0)$ for $m = 0, \ldots, k$, where we used (4·6) for $m = k$. Since $\lambda^{(m)}(p) = 0$ $(m \leqslant k-1)$ and $\lambda^{(k)}(p) \neq 0$, (4·5) yields the first part of the lemma.

Next, we show the second part: Differentiating (4·8) with respect to z_j

$$\Phi''_{z_j} = \langle \nu_{z_j}, f - (X, Y) \rangle_K + \langle \nu'', f_{z_j} - \boldsymbol{e}_{j+2} \rangle_K + \frac{\partial}{\partial z_j} \langle \nu', f' \rangle_K,$$

where $\boldsymbol{e}_{j+2}$ is defined in (3·2). As in (4·9), applying Lemma 3·2, we have

$$\Phi^{(m)}_{z_j}(\mathbf{0}, 0) = a\lambda^{(m-2)}_{z_j} = \begin{cases} 0 & (j > m-1),\\ a(p)\lambda^{(m-2)}_{z_{m-1}}(p) \neq 0 & (j = m-1).\end{cases}$$

Since $\Phi_{x_j} = \langle \nu, -\boldsymbol{e}_j \rangle_K = -\nu^j$ for $j = 1, 2$, the Jacobian matrix of $(\Phi, \Phi', \ldots, \Phi^{(k)})$ at the origin is written as

$$\begin{pmatrix} -\nu^1 & -\nu^2 & 0 & 0 & \ldots & 0 & \ldots & 0\\ -(\nu^1)' & -(\nu^2)' & 0 & 0 & \ldots & 0 & \ldots & 0\\ * & * & a\lambda_{z_1} & 0 & \ldots & 0 & \ldots & 0\\ * & * & * & a\lambda'_{z_2} & \ldots & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ * & * & * & * & * & a\lambda^k_{z_{k-1}} & \ldots & 0 \end{pmatrix},$$

which is of rank $k+1$.

Proof of Theorem 2·4 It can be straightforwardly checked that the singular point given in (1·1) is k-nondegenerate, but not $(k+1)$-singular. We now prove the converse. By Zakalyukin's theorem [**11**] (see also the appendix of [**7**]), $\mathrm{Im}(f)$ is locally diffeomorphic to the standard A_{k+1}-singularity if and only if it is $\boldsymbol{K}$-right-left equivalent to (1·1), whenever the regular points of f are dense. Thus, it is sufficient to show the versality of Φ (which implies that D_Φ is locally diffeomorphic to the standard A_{k+1}-singularity, see [**4**]). In fact, this is evident by Lemma 4·3.

Remark. In [**7**], versal unfoldings of a cuspidal edge and a swallowtail in $\boldsymbol{R}^3$ are given. The above versal unfolding Φ is much simpler than those in [**7**].

Now, we give proofs of Corollaries 2·5, 2·6 and 2·8 using Theorem 2·4.

Proof of Corollary 2·5. The necessary part is obvious by Lemma 3·2. We assume that $\lambda = \lambda' = \cdots = \lambda^{(k)} = 0$, $\lambda^{(k+1)} \neq 0$ at p and the map $(\lambda, \lambda', \ldots, \lambda^{(k)})$ is non-singular at p. Then we can see that p is 1-nondegenerate, so we can take the 1-st adopted coordinate system. By Lemma 3·2, p is 2-singular if $k \geqslant 2$. Since (λ, λ') is full rank, p is 2-nondegenerate.

We can continue this argument k times. Then we see that p is k-nondegenerate but not $(k+1)$-singular. By Theorem 2·4, we have the conclusion.

Proof of Corollary 2·6. By Lemma 2·3, p is 2-singular if and only if $\eta \in T_pS(f) = \mathrm{Span}_{\boldsymbol{K}}\{v_1, \ldots, v_{n-1}\}$, which is equivalent to $\mu(p) = 0$. Hence p is A_2-singular point if and only if $\mu(p) \neq 0$.

Since the condition does not depend on the choice of coordinate system and a frame $\{v_1, \ldots, v_{n-1}\}$ on $S(f)$, we take a coordinate system $(x_1, \ldots, x_n)$ of U such that $x_n = \lambda$ and set $v_j = \partial/\partial x_j$ $(j = 1, \ldots, n-1)$. Then the singular set $S(f)$ is given as a hyperplane $x_n = 0$, and $\mu = \eta_n$ holds on $S(f)$, where $\eta = (\eta_1, \ldots, \eta_n)$. On the other hand, $\lambda' = d\lambda(\eta) = dx_n(\eta) = \eta_n$, and then we have $\lambda' = \mu$ on $S(f)$. Then, if p is 2-singular, it is 2-nondegenerate if and only if $d\lambda'(T_pS(f)) = d\mu(T_pS(f)) \neq 0$. Moreover, since

$$S_2 = \{q \in S(f)\,;\ \lambda'(q) = 0\} = \{q \in S(f)\,;\ \mu(q) = 0\},$$

p is 3-singular if and only if $d\mu(\eta_p) = 0$. Thus, we have a criterion for A_3-singularities.

Proof of Corollary 2·8. Since p is 2-nondegenerate, $d\mu_p \neq 0$ holds by Corollary 2·6. Hence, in addition to the previous proof, one can take a coordinate system $(x_1, \ldots, x_n)$ on U such that $\lambda = x_n$ and $\mu = x_{n-1}$ on $S(f)$. Then the set S_2 is a linear subspace $\{(x_1, \ldots, x_{n-2}, 0, 0)\}$ near p. As seen in the previous proof, $\lambda' = \mu$ holds on $S(f)$, and hence the condition (2·5) in Corollary 2·5 is equivalent to (2·7) under the assumption that p is 2-singular. Moreover, since $\lambda'' = \mu'$ on S_2, the Jacobian matrix of the map Λ in Corollary 2·5 is computed as

$$\begin{pmatrix} \lambda_{x_1} & \cdots & \lambda_{x_{n-2}} & \lambda_{x_{n-1}} & \lambda_{x_n} \\ \lambda'_{x_1} & \cdots & \lambda'_{x_{n-2}} & \lambda'_{x_{n-1}} & \lambda'_{x_n} \\ \lambda''_{x_1} & \cdots & \lambda''_{x_{n-2}} & \lambda''_{x_{n-1}} & \lambda''_{x_n} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \lambda^{(k-1)}_{x_1} & \cdots & \lambda^{(k-1)}_{x_{n-2}} & \lambda^{(k-1)}_{x_{n-2}} & \lambda^{(k-1)}_{x_n} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 & 0 & 1 \\ 0 & \cdots & 0 & 1 & * \\ \mu'_{x_1} & \cdots & \mu'_{x_{n-2}} & * & * \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \mu^{(k-1)}_{x_1} & \cdots & \mu^{(k-1)}_{x_{n-2}} & * & * \end{pmatrix}$$

holds at p. Hence the conclusion follows.

Example 4·4 (Tangent developables in $\boldsymbol{K}^4$). Let $\gamma(z)$ be a $\boldsymbol{K}$-differentiable curve in $\boldsymbol{K}^4$ such that $\gamma', \gamma'', \gamma''', \gamma''''$ are linearly independent. We consider a hypersurface $f(z, u, v) = \gamma(z) + u\gamma'(z) + v\gamma''(z)$. If we set $\nu = \gamma' \wedge \gamma'' \wedge \gamma'''$, then f is a front with $\boldsymbol{K}$-normal vector ν. Furthermore, we have

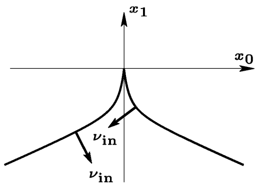
$$S_1 = \{v = 0\}, \quad S_2 = \{u = v = 0\}, \quad \eta = (-1, 1, u) \quad \text{and} \quad \lambda = v.$$

By Corollary 2·5, f at $(z, u, 0)$ is $\boldsymbol{K}$-right-left equivalent to (cusp) $\times\, \boldsymbol{K}^2$ if and only if $u \neq 0$, and f at $(z, 0, 0)$ is $\boldsymbol{K}$-right-left equivalent to (swallowtail) $\times\, \boldsymbol{K}$.

5. *Zigzag numbers on A-fronts*

Let M^n (resp. N^{n+1}) be a manifold of dimension n (resp. $n+1$), and $f\colon M^n \to N^{n+1}$ a front. If all singularities of f are A_k-singular points $(k \leqslant n+1)$, it is called an *A-front*. In this section, we set $\boldsymbol{K} = \boldsymbol{R}$, and discuss a topological invariant of A-fronts. Now, we assume that M^n is orientable.

For each A_k $(k \leqslant n+1)$ singular point on an A-front, the image of it coincides with the discriminant set $\{F = F_t = 0\}$ of (1·2). Then we can define the *inward normal vector* ν_{in} which points in the direction that the number of roots of polynomial $F(t) = 0$ increasing.



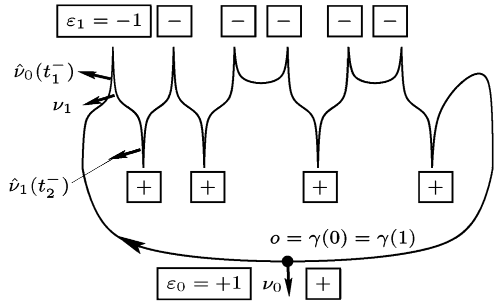


Fig. 1. Zigzag number of a plane curve.

For example, on the A_2-singularity $(t, x_2, \ldots, x_n) \mapsto (2t^3, -3t^2, x_2, \ldots, x_n)$, ν_{in} is defined by

$$\nu_{\text{in}} = \begin{cases} -(1, t, 0, \ldots, 0) & (t > 0), \\ (1, t, 0, \ldots, 0) & (t < 0), \end{cases}$$

see Figure 1, left. Then, the inward direction is uniquely defined on a neighborhood of each connected component of $S(f)$. On the other hand, a front $f \colon M^n \to N^{n+1}$ is called a *Morin front* if there exists a C^∞-map $\hat{f}$ from an $(n+1)$-manifold into N^{n+1} such that:

(i) $\hat{f}$ admits only Morin singularities;
(ii) f is the restriction of $\hat{f}$ to the singular set $S(\hat{f})$.

Morin fronts are all A-fronts (see Corollary 2·11 and the Appendix). Conversely, any A-fronts are locally Morin fronts. The inward direction of the Morin front f points in the direction of the image of $\hat{f}$, and we get the following:

PROPOSITION 5·1. *A Morin front has a unique unit normal vector on the set of regular points which points in the inward direction.*

We fix a regular point $o \in M^n$ of f. A continuous map $\gamma \colon [0, 1] \longrightarrow M^n$ is called a *loop* at o if it satisfies $\gamma(0) = \gamma(1) = o$. We fix a Riemannian metric g of N^{n+1}. A loop γ is called *co-orientable* if there exists a unit normal vector field $\nu(t) \in T_{\hat{\gamma}(t)} N^{n+1}$ of f along γ such that $\nu(0) = \nu(1)$, where $\hat{\gamma}(t) := f \circ \gamma(t)$. If γ is not co-orientable, it is called *non-co-orientable*. These two properties of loops depend only on their homotopy classes. We set

$$\rho_f(\gamma) := \begin{cases} 0 & (\text{if } \gamma \text{ is co-orientable}), \\ 1 & (\text{if } \gamma \text{ is non-co-orientable}). \end{cases}$$

Then it induces a representation $\rho_f \colon \pi_1(M^n) \longrightarrow \mathbf{Z}_2 = \mathbf{Z}/2\mathbf{Z}$. Next, we set

$$\hat{\pi}_1(M^n) := \ker(\rho_f)\big(\subset \pi_1(M^n)\big)$$

which consists of homotopy classes of co-orientable loops. We shall now construct the secondary homotopical representation $\hat{\rho}_f \colon \hat{\pi}_1(M^n) \longrightarrow \mathbf{Z}$ called the *Maslov representation* as follows: Let $f \colon M^n \to N^{n+1}$ be a front and

$$S(f) = \bigcup_{\alpha \in \Lambda} \Gamma_\alpha$$

be the decomposition of the singular set $S(f)$ by its connected components. Then there exists a disjoint family of open domains O_α in M^n such that $\Gamma_\alpha \subset O_\alpha$. Let $\gamma\colon [0,1] \to M^n$ be a co-orientable loop at $o \notin S(f)$. We set

$$S_\gamma := \{t \in [0,1]\,;\ \gamma(t) \in S(f)\}.$$

For each $t \in S_\gamma$, there exists an open interval $I_t(\subset (0,1))$ such that $\gamma(I_t) \subset O_\alpha$ for some $\alpha \in \Lambda$. Since S_γ is compact, there exists a finite number of open intervals $I_1 = (t_1^-, t_1^+), \ldots, I_m = (t_m^-, t_m^+)$ such that $S_\gamma \subset I_1 \cup \cdots \cup I_m$. If $I_k \cap I_{k+1}$ is non-empty, we can replace I_k and I_{k+1} by $I_k \cup I_{k+1}$. After these operations, each I_k is still contained in some O_α, since $\{O_\alpha\}_{\alpha\in\Lambda}$ is disjoint. Then we may assume that

$$0 < t_1^- < t_1^+ < t_2^- < \cdots < t_m^- < t_m^+ < 1.$$

We fix a unit normal vector $\nu_0 \in T_{\hat{\gamma}(o)} N^{n+1}$ at o. (If f admits a globally defined smooth unit normal vector field ν on M^n, one of the canonical choices is $\nu_0 = \nu(0)$.) Let ν_j $(j = 1, 2, \ldots, m)$ be the inward unit normal vector at $t = t_j^+$. Then we can take a continuous unit vector field $\hat{\nu}_j(t)$ $(t_j^+ \leqslant t \leqslant t_{j+1}^-)$ along $\gamma|_{[t_j^+, t_{j+1}^-]}$ such that

$$\hat{\nu}_i(t_j^+) = \nu_j \qquad (i = 0, 1, 2, \ldots, m),$$

where $t_0^+ = 0$ and $t_{m+1}^- = 1$. We set $\varepsilon_0 = +1$ and

$$\varepsilon_j := \begin{cases} \varepsilon_{j-1} & (\text{if } \hat{\nu}_{j-1}(t_j^-) \text{ is inward}) \\ -\varepsilon_{j-1} & (\text{if } -\hat{\nu}_{j-1}(t_j^-) \text{ is inward}) \end{cases} \qquad (j = 1, 2, \ldots, m+1),$$

where $\nu_{m+1} = \nu_0$. Then we get a sequence

$$\varepsilon_0,\ \varepsilon_1,\ \varepsilon_2, \ldots, \varepsilon_m,\ \varepsilon_{m+1}. \tag{5·1}$$

Since M^n is orientable, m is an odd integer. By the co-orientability of γ, $\varepsilon_0 = \varepsilon_{m+1} = +1$ holds. For simplicity, we write $\varepsilon_j = +$ (resp. $-$) if $\varepsilon_j = +1$ (resp. -1). If there are two successive same signs $++$ or $--$, we cancel them. Repeating this cancellation, we get a sequence which has no successive signs

$$+ - \cdots + - +\,.$$

We shall call this reduction the *normalization* of the sequence (5·1). Then the number of – after the normalization is called the *zig-zag number* of the co-orientable loop γ. (The definition of zig-zag number for surfaces in $\boldsymbol{R}^3$ is given in [**8**]. See also [**10**].) For example, we set $M = S^1$ and $N = \boldsymbol{R}^2$ and consider a front as in Figure 1, right. Then this closed curve induces a sequence

$$+ - + - + - - + - - ++$$

which can be reduced to

$$+ - +(-(+(--)+)-) - (++) \quad \text{that is} \quad + - + -$$

and thus the zig-zag number in this case is 2. We denote by $z(\gamma)$ the zig-zag number of the co-orientable loop γ. (Global properties and references of zig-zag numbers of planar curves are given in [**1**].)

PROPOSITION 5·2. *The zig-zag number $z(\gamma)$ depends only on the homotopy class of γ, and is stable under the deformation of an A-front.*

Proof. Suppose $[\gamma_0]=[\gamma_1]$ in $\pi_1(M^n)$. Then there exists a homotopy $\Gamma\colon [0,1]\times[0,1]\to M^n$ such that $\Gamma(t,0)=\gamma_0(t)$ and $\Gamma(t,1)=\gamma_1(t)$. We set

$$S_\Gamma := \{(t,s)\in[0,1]\times[0,1];\ \Gamma(t,s)\in S(f)\}.$$

For each $(t,s)\in S_\Gamma$, there exists an open neighbourhood $W_{t,s}\subset[0,1]\times[0,1]$ of (t,s) such that $\Gamma(W_{t,s})\subset O_\lambda$ for some $\lambda\in\Lambda$. Since $[0,1]\times[0,1]$ is compact, it is covered by just finitely many such open subsets. Then the standard homotopy argument yields the homotopy invariance $z(\gamma_0)=z(\gamma_1)$.

A closed C^1-regular curve $\gamma : [0,1]\to M^n$ starting at o is called a *null loop* if $\gamma'(t)$ ($t\in[0,1]$) belongs to the kernel of df whenever $\gamma(t)\in S(f)$. One can easily show that, for each homotopy class of the fundamental group $\pi_1(M^n)$, there exists a null C^1-loop which represents the homotopy class. Moreover, we may assume that $\gamma(t)$ passes through only finitely many A_2-singular points. Suppose that $\gamma(t)$ is a co-orientable null loop. Then there exists a continuous unit normal vector field $\nu(t)$ along $\gamma(t)$ such that $\nu(0)=\nu_0$. Since γ is co-orientable, $\nu(1)=\nu_0$ holds. As in [**10**], we define a continuous map of $[0,1]\setminus S_\gamma$ into $P^1(\boldsymbol{R})$ by

$$C_\gamma\colon t\longmapsto [g(\hat\gamma'(t),\hat\gamma'(t)) : g(\nu_\gamma'(t),\hat\gamma'(t))],$$

which can be extended continuously across S_γ. We call this map $C_\gamma : [0,1]\to \boldsymbol{R}P^1$ *the normal curvature map*. In fact, when $\gamma(t)$ is a regular point of f, $-g(\nu_\gamma',\hat\gamma')/g(\hat\gamma',\hat\gamma')$ is exactly the normal curvature of f along γ. This is independent of the orientation of γ, but its sign depends on the initial choice of the unit normal vector $\pm\nu_0$ at o. We denote the rotation index of the map C_γ by $\mu_f(\gamma)$. The absolute value of $\mu_f(\gamma)$ is called the *Maslov index* of the co-orientable null loop γ. Then by the same argument as in [**10**], we get the following:

PROPOSITION 5·3. *The Maslov index $|\mu_f(\gamma)|$ coincides with the zig-zag number $z(\gamma)$. In particular, we get a* (*Maslov*) *representation*

$$\hat\rho_f\colon \hat\pi_1(M^n)\ni[\gamma]\longmapsto \mu_f(\gamma)\in\mathbf{Z}.$$

Since the inward direction is defined globally for a Morin front, we have the following:

COROLLARY 5·4. *The Maslov representations for Morin fronts are trivial.*

Remark 5·1. Suppose that M^n and N^{n+1} are both oriented and f is co-orientable. Then, we can take a globally defined unit normal vector field ν along f which gives a direct interpretation of the sign ε_j: We denote by O_α^+ the component of $O_\alpha\setminus\Gamma_\alpha$ where v is compatible with the orientation of M^n. Then each component Γ_α of $S(f)$ is called *zig* if v coincides with v_{in} on O_α^+. Some global properties of zig-zag numbers are given in [**8**].

Appendix A. Criteria of Morin type singularities

The *A_k-Morin singularities* are map germs which are $\boldsymbol{K}$-right-left equivalent to

$$f(z_1,\ldots,z_n)=\left(z_1z_n+\cdots+z_{k-1}z_n^{k-1}+z_n^{k+1},z_1,\ldots,z_{n-1}\right)\qquad (k\leqslant n)\qquad \text{(A 1)}$$

at $\mathbf{0}$ ([**9**], see also [**5**]). The A_0-Morin singularity means a regular point. The following assertion holds:

THEOREM A1. *Assume that $k \leqslant n$. Let U be a domain in $\boldsymbol{K}^n$, and $f\colon U \to \boldsymbol{K}^n$ a $\boldsymbol{K}$-differentiable map and $p \in S(f)$. We assume that p is a corank one singularity. Then f at p is $\boldsymbol{K}$-right-left equivalent to A_k-Morin singularity if and only if*

$$\lambda = \lambda' = \cdots = \lambda^{(k-1)} = 0,\ \lambda^{(k)} \neq 0 \text{ at } p \quad \text{and} \tag{A 2}$$

$$\Lambda := \big(\lambda, \lambda', \dots, \lambda^{(k-1)}\big) : U \to \boldsymbol{K}^k \quad \text{is non-singular at } p. \tag{A 3}$$

Here, $\lambda = \det(f_{x_1}, \dots, f_{x_n})$, $(x_1, \dots, x_n)$ is the canonical coordinate system on U, $\lambda' = \tilde{\eta}\lambda (= \lambda^{(1)})$, $\lambda^{(i)} = \tilde{\eta}\lambda^{(i-1)}$ and $\tilde{\eta}$ is the extended null vector field of f.

Proof. Since the conditions (A 2) and (A 3) do not change when changing λ to $\psi\lambda$, where $\psi(p) \neq 0$, the conditions (A 2) and (A 3) are independent on the choice of local coordinate systems on the source and target. Suppose f has the form

$$f(z_1, \dots, z_n) = \big(\mu(z_1, \dots, z_n), z_1, \dots, z_{n-1}\big), \tag{A 4}$$

such that:

(i) $\mu' = \cdots = \mu^{(k)} = 0$, $\mu^{(k+1)} \neq 0$ at p ($' = \partial/\partial z_n$); and
(ii) the map germ of $(\mu', \dots, \mu^{(k)})$ is nonsingular at p.

Then f is an A_k-Morin singularity (see [**5**, page 177]). Though the proof of this fact in [**5**] is given for $\boldsymbol{K} = \boldsymbol{R}$, the argument works also for the case of $\boldsymbol{K} = \boldsymbol{C}$. By (A 4), we have $\lambda = \mu'$, and $\eta = \partial_{z_n}$ holds. This implies the theorem.

The same type of assertion as in Corollaries 2·6 and 2·8 hold for A_k-Morin singularities.

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