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Self-duality of Selmer groups

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Abstract

The first part of the paper gives a new proof of self-duality for Selmer groups: if A is an abelian variety over a number field K , and F/K is a Galois extension with Galois group G , then the $\mathbb{Q}_p G$ -representation naturally associated to the p^∞ -Selmer group of A/F is self-dual. The second part describes a method for obtaining information about parities of Selmer ranks from the local Tamagawa numbers of A in intermediate extensions of F/K .

1. Introduction

Let F/K be a Galois extension of number fields with Galois group G , and A an abelian variety defined over K . The action of G on the F -rational points of A defines a $\mathbb{Q}G$ -representation $A(F) \otimes \mathbb{Q}$, which in particular recovers the Mordell–Weil ranks of A over all intermediate extensions.

Now if p is a prime number, there is an analogous picture for Selmer groups. Let

$$\mathcal{X} = \mathcal{X}_p(A/F) = (\text{Pontryagin dual of the } p^\infty\text{-Selmer group of } A/F) \otimes \mathbb{Q}_p.$$

It is a \mathbb{Q}_p -vector space whose dimension is the p^∞ -Selmer rank $\text{rk}_p(A/F)$, the Mordell–Weil rank plus the number of copies of $\mathbb{Q}_p/\mathbb{Z}_p$ in the Tate–Shafarevich group $\text{III}(A/F)$. Moreover, it is a G -representation, and it also recovers $\text{rk}_p(A/L)$ for intermediate extensions

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as the dimension of $\mathcal{X}^{\text{Gal}(F/L)}$. In view of the conjectural finiteness of the Tate-Shafarevich group, \mathcal{X} and $A(F) \otimes \mathbb{Q}_p$ should be isomorphic as G -representations.

Decompose $\mathcal{X} \cong \bigoplus_{\rho} \rho^{\oplus m_{\rho}}$ into $\mathbb{Q}_p G$ -irreducible constituents. The structure of \mathcal{X} is encoded in the multiplicities m_{ρ} . Our main result (Theorem 1.6 below) may be stated as

$$\sum_{\rho \in S_{\Theta}} m_{\rho} \pmod{2} = (\text{explicit local data})$$

for certain special sets S_{Θ} of representations.

A result of this type is proved in [3] for Mordell–Weil groups under the assumption that $\text{III}(A/F)$ is finite, and it relies crucially on the existence of the height pairing. The Selmer group also admits a non-degenerate G -invariant pairing

$$\langle \cdot, \cdot \rangle: \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{Q}_p.$$

Equivalently,

THEOREM 1.1. *\mathcal{X} is self-dual as a $\mathbb{Q}_p G$ -representation.*

This is essentially a consequence of Poitou–Tate duality, and it may be deduced using the methods of Greenberg, see [6, proposition 2]. A proof in the general context of Bloch–Kato Selmer groups is given by Nekovář in [15, 12.5.9.5(iv)]. We present an alternative proof in Section 2.

Let us briefly record two straightforward consequences of self-duality, whose proofs we postpone to Section 2. The first of these relies on the parity conjecture for elliptic curves over \mathbb{Q} , which is known thanks to the work of Birch–Stephens [1], Greenberg [5] and Guo [8] (E CM), Monsky [14] ($p = 2$), Nekovář [15] (p potentially ordinary), Kim [9] (p supersingular) and [3] (p odd). In particular, Theorem 1.2 is proved in [15] when E has potentially ordinary reduction at p .

THEOREM 1.2 (= Theorem 2.8). *Let E/\mathbb{Q} be an elliptic curve. For every abelian extension F/\mathbb{Q} and every prime p ,*

$$\text{rk}_p(E/F) \equiv \text{ord}_{s=1} L(E/F, s) \pmod{2}.$$

THEOREM 1.3 (= Corollary 2.5). *Let A/K be an abelian variety, and suppose F/K is Galois of odd degree. Then $\text{rk}_p(A/F) \equiv \text{rk}_p(A/K) \pmod{2}$.*

Returning to the representation-theoretic structure of \mathcal{X} , suppose we are given a relation between induced representations (fixed for the rest of the introduction),

$$\Theta: \bigoplus_i \text{Ind}_{H_i}^G \mathbf{1} \cong \bigoplus_j \text{Ind}_{H'_j}^G \mathbf{1} \quad (H_i, H'_j < G).$$

Observe that Artin formalism forces an equality of L -functions

$$\prod L(A/F^{H_i}, s) = \prod L(A/F^{H'_j}, s),$$

and that the conjectural Birch–Swinnerton-Dyer formula at $s = 1$ implies a relation between the arithmetic invariants of A over these fields. As explained in [3, section 2.2] most of these cancel modulo rational squares, leading to

CONJECTURE 1.4 (\square -Conjecture). *Suppose A has a principal polarisation induced by a K -rational divisor (e.g. A is an elliptic curve.) Then*

$$\frac{\prod_i \text{Reg}(A/F^{H_i})}{\prod_j \text{Reg}(A/F^{H_j})} \equiv \frac{\prod_i C(A/F^{H_i})}{\prod_j C(A/F^{H_j})} \pmod{\mathbb{Q}^{*2}}.$$

Here Reg is the regulator, and C is a product of local terms (essentially Tamagawa numbers, see the list of notation below). The assumption on A guarantees that III modulo its divisible part has square order.

We showed ([3, corollary 2.5]) that this conjecture is implied by the Shafarevich-Tate conjecture on the finiteness of III . Now we use the pairing \langle, \rangle on \mathcal{X} to prove the corresponding unproven statement for Selmer groups. For every subgroup H of G define

$$\text{Reg}_p^{(\cdot)}(A/F^H) = \det \left(\frac{1}{|H|} \langle, \rangle \mid \mathcal{X}^H \right) \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2},$$

computing the determinant on any basis of \mathcal{X}^H . This, including the scaling, is analogous to the definition of $\text{Reg}(A/F^H)$ as the determinant of the height pairing on $A(F^H)$ modulo torsion.

THEOREM 1.5 (=Theorem 3.1). *Suppose A/K is principally polarised. If $p = 2$, assume furthermore that the principal polarisation on A is induced by a K -rational divisor. Then*

$$\text{ord}_p \frac{\prod_i \text{Reg}_p^{(\cdot)}(A/F^{H_i})}{\prod_j \text{Reg}_p^{(\cdot)}(A/F^{H_j})} \equiv \text{ord}_p \frac{\prod_i C(A/F^{H_i})}{\prod_j C(A/F^{H_j})} \pmod{2}.$$

The theorem may be used to express parities of Selmer ranks in terms of local invariants. The crucial point is that the left-hand side depends only on \mathcal{X} as a $\mathbb{Q}_p G$ -representation, and not on the pairing \langle, \rangle . Let \mathcal{S} be the set of self-dual $\mathbb{Q}_p G$ -representations, which are either irreducible or of the form $T \oplus T^*$ for some irreducible $T \not\cong T^*$ (T^* is the contragredient of T). Any self-dual $\mathbb{Q}_p G$ -representation can be uniquely decomposed into such constituents. For every $\rho \in \mathcal{S}$, pick a non-degenerate G -invariant pairing $\langle\langle, \rangle\rangle$ on ρ , and define the regulator constant

$$\mathcal{C}(\Theta, \rho) = \frac{\prod_i \det \left(\frac{1}{|H_i|} \langle\langle, \rangle\rangle \mid \rho^{H_i} \right)}{\prod_j \det \left(\frac{1}{|H_j|} \langle\langle, \rangle\rangle \mid \rho^{H_j} \right)} \in \mathbb{Q}_p^*/\mathbb{Q}_p^{*2}.$$

Consider the set

$$\mathcal{S}_\Theta = \{ \rho \in \mathcal{S} \mid \text{ord}_p \mathcal{C}(\Theta, \rho) \equiv 1 \pmod{2} \}.$$

This is a “computable combination of representations” in the following sense:

THEOREM 1.6. *Suppose A/K is principally polarised, and if $p = 2$ assume furthermore that the polarisation is induced by a K -rational divisor. There is a decomposition $\mathcal{X} \cong \bigoplus_{\rho \in \mathcal{S}} \rho^{\oplus m_\rho}$, and*

$$\sum_{\rho \in \mathcal{S}_\Theta} m_\rho \equiv \text{ord}_p \frac{\prod_i C(A/F^{H_i})}{\prod_j C(A/F^{H_j})} \pmod{2}.$$

Proof. By self-duality, such a decomposition exists. Take the obvious pairing \langle, \rangle on \mathcal{X} coming from $\langle\langle, \rangle\rangle$ on its constituents and apply Theorem 1.5.

Note that in practice the right-hand side is very explicit: it can be computed for elliptic curves by Tate’s algorithm ([16, IV.9]), for semistable abelian varieties from the monodromy pairing ([7, section 10]), and for Jacobians of curves using the intersection pairing ([2, section 1]).

In Examples 3.5–3.7 we will illustrate this theorem for specific relations when $G = D_{2p}, C_p \rtimes C_{p-1}$ and $GL_2(\mathbb{F}_p)$. The first two were already considered in [3], but required ad hoc constructions of isogenies in absence of Theorem 1.6, see [3, Theorem 4.11, proposition 4.17]. The D_{2p} -extensions were also studied by Mazur and Rubin, who give another local expression for the same parity of Selmer ranks, see [11, Theorem A]. We do not have an intrinsic description of the sets \mathcal{S}_Θ for a general group G .

In the forthcoming paper [4] we will expand on the properties of permutation relations and regulator constants, and address the question of the compatibility of Theorem 1.6 with root numbers. Under reasonably mild hypotheses on A (e.g. when A is an elliptic curve whose additive primes above 2 and 3 are unramified in F/K), we will show that the parity of $\sum_{\rho \in \mathcal{S}_\Theta} m_\rho$ is indeed determined by the (conjectural) sign in the functional equation of a corresponding L -function, as predicted by the parity conjecture.

Notation. Throughout the paper we fix:

- F/K Galois extension of number fields;
- G $\text{Gal}(F/K)$;
- A/K abelian variety with a fixed regular non-zero exterior form ω ;
- p prime number.

For an intermediate field $K \subset L \subset F$, we use the following notation:

- $\text{Sel}_{p^\infty}(A/L)$ the p^∞ -Selmer group $\varinjlim \text{Sel}_{p^n}(A/L)$;
- $X_p(A/L)$ Pontryagin dual $\text{Hom}(\text{Sel}_{p^\infty}(A/L), \mathbb{Q}_p/\mathbb{Z}_p)$ modulo torsion;
- $\mathcal{X}_p(A/L)$ $X_p(A/L) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$;
- $\text{rk}_p(A/L)$ p^∞ -Selmer rank of A/L , i.e. $\dim \mathcal{X}_p(A/L)$;
- $C(A/L)$ $\prod c_v |\omega/\omega_v^o|_v$, where the product is taken over all primes of L , c_v is the local Tamagawa number, ω_v^o the Néron differential and $|\cdot|_v$ the normalised absolute value.

By convention, permutation modules $\mathbb{Q}[G/H] \cong \text{Ind}_H^G \mathbf{1}$ come with a standard basis of elements of G/H . With respect to this basis, the identity matrix defines a G -invariant pairing.

2. Self-duality

The purpose of this section is to prove:

THEOREM 2.1 (= Theorem 1.1). *Suppose F/K is a Galois extension of number fields with Galois group G , A/K an abelian variety, and p a prime number. Then $\mathcal{X}_p(A/F)$ is self-dual as a $\mathbb{Q}_p G$ -representation.*

To begin with, if M is a finitely generated $\mathbb{Z}G$ -module, free over \mathbb{Z} , there is a naturally associated abelian variety $A \otimes M$ over K ([12, section 2]). When L/K is an intermediate extension, $A \otimes \mathbb{Z}[G/\text{Gal}(F/L)]$ is the Weil restriction of scalars $W_{L/K}A$. An injection of $\mathbb{Z}G$ -modules $\phi: M_1 \rightarrow M_2$ with finite cokernel induces a K -isogeny $f_\phi: A \otimes M_1 \rightarrow A \otimes M_2$. If A is principally polarised and M_1 and M_2 are permutation modules, then $A \otimes M_i$ carry induced polarisations, with respect to which the dual isogeny $f_\phi^t: (A \otimes M_2)^t \rightarrow (A \otimes M_1)^t$ comes from the transposed matrix ϕ^t in the standard bases ([3, section 4.2]). In other words, $f_{\phi^t} = (f_\phi)^t$.

An isogeny $f: A \rightarrow B$ induces a G -invariant map $X_p(B/K) \rightarrow X_p(A/K)$, which is an isomorphism when tensored with \mathbb{Q}_p . Following [3, section 4], write

$$Q(f) = |\text{coker}(f: A(K)/A(K)_{\text{tors}} \rightarrow B(K)/B(K)_{\text{tors}})| \times |\ker(f: \text{III}(A)_{\text{div}} \rightarrow \text{III}(B)_{\text{div}})|,$$

where III_{div} denotes the divisible part of III . Then $Q(f)$ is multiplicative under composition of isogenies, and its p -part is the size of the cokernel of $f: X_p(B/K) \rightarrow X_p(A/K)$.

Recall the Selmer group analogue of the invariance of the Birch–Swinnerton-Dyer quotient under isogenies:

THEOREM 2.2. ([3, Theorem 4.3]) *Let $A, B/K$ be abelian varieties given with regular non-zero exterior forms ω_A, ω_B , and suppose $\phi: A \rightarrow B$ is a K -isogeny. Writing $\text{III}_0(A/K)$ for $\text{III}(A/K)$ modulo its divisible part and*

$$\Omega_A = \prod_{\substack{v|\infty \\ \text{real}}} \int_{A(K_v)} |\omega_A| \cdot \prod_{\substack{v|\infty \\ \text{complex}}} 2 \int_{A(K_v)} \omega_A \wedge \bar{\omega}_A$$

and similarly for B , we have

$$\frac{|B(K)_{\text{tors}}| |B'(K)_{\text{tors}}| C(A/K) \Omega_A}{|A(K)_{\text{tors}}| |A'(K)_{\text{tors}}| C(B/K) \Omega_B} \prod_{l|\deg \phi} \frac{|\text{III}_0(A)[l^\infty]|}{|\text{III}_0(B)[l^\infty]|} = \frac{Q(\phi')}{Q(\phi)}.$$

To prove Theorem 2.1, we establish the analogous statement for $\mathcal{X}_p(A \otimes \mathbb{Z}[G]/K)$ in place of $\mathcal{X}_p(A/F)$, and then show that the two are isomorphic.

THEOREM 2.3. *Let F/K be a Galois extension of number fields, A/K an abelian variety and p a prime number. Then $\mathcal{X}_p(A \otimes \mathbb{Z}[G])$ is a self-dual $\text{Gal}(F/K)$ -representation.*

Proof. Let $M = \mathbb{Z}[G]$ and $\mathcal{X} = \mathcal{X}_p(A \otimes M/K)$. The idea is that for a self-isogeny

$$f: A \otimes M \rightarrow A \otimes M$$

we have $Q(f) = Q(f')$ by Theorem 2.2. We will construct an f whose Q recovers the multiplicity of a given representation in \mathcal{X} , and $Q(f')$ recovers the multiplicity of its dual.

To begin with, after passing to an isogenous abelian variety if necessary, we may assume that A is principally polarised. We have to show that for every $g \in G$, its eigenvalues on \mathcal{X} come in inverse pairs. Restricting to the subgroup generated by g , we will also assume that G is cyclic.

Let τ_i be the distinct \mathbb{Q}_p -irreducible representations of G and write $m_{\tau_i}(\mathcal{X})$ for their multiplicity in \mathcal{X} . Choose G -invariant \mathbb{Z}_p -sublattices Λ_{τ_i} of $X_p(A \otimes M/K)$ with $\Lambda_{\tau_i} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \tau_i^{\oplus m_{\tau_i}(\mathcal{X})}$, so $\bigoplus \Lambda_{\tau_i}$ is of finite index in $X_p(A \otimes M/K)$.

For $\tau \in \{\tau_i\}$, let $P_\tau = \sum_{g \in G} \text{Tr}(\tau(g))g^{-1} \in \mathbb{Z}_p[G]$ be “ $|G|$ times the projector onto the τ component” operator. Elements in an open neighbourhood U of $|G| + (p-1)P_\tau$ act as $|G|$ times an isomorphism on all Λ_{τ_i} save Λ_τ , where they act by $p|G|$ times an isomorphism. Similarly there is such a neighbourhood U^* for τ^* . Since the \mathbb{Q}_p -linear map on $\mathbb{Q}_p[G]$ determined by $g \rightarrow g^{-1}$ for $g \in G$ is continuous and sends P_τ to P_{τ^*} , we can choose $\Phi = \sum_g x_g g \in \mathbb{Z}[G] \cap U$ with $\Phi^* = \sum_g x_g g^{-1} \in U^*$.

Since G is commutative, Φ defines a G -endomorphism ϕ of M . Considering its action on the Λ_{τ_i} ,

$$Q(f_\phi) = Q(f|_G) p^{m_\tau(\mathcal{X}) \dim \tau}.$$

Also, Φ^* corresponds to ϕ^t in $\text{End}(M)$ with respect to the standard basis of $M = \mathbb{Z}[G]$, so

$$Q(f_{\phi^t}) = Q(f_{|G|}) p^{m_{\tau^*}(\mathcal{X}) \dim \tau^*}.$$

Since $Q(f_\phi)/Q(f_{\phi^t}) = 1$ by Theorem 2.2, $m_\tau(\mathcal{X}) = m_{\tau^*}(\mathcal{X})$. As this holds for all τ , the asserted self-duality follows.

LEMMA 2.4. *Let A/K be an abelian variety, and W the Weil restriction $W_{F/K}A$. There is a canonical isomorphism of G -modules*

$$\text{Sel}_{p^n}(W/K) \longrightarrow \text{Sel}_{p^n}(A/F),$$

where G acts on $\text{Sel}_{p^n}(W/K)$ via automorphisms of W/K and on $\text{Sel}_{p^n}(A/F)$ by its usual action on $H^1(F, A[p^n])$. In particular, $\mathcal{X}_p(W/K) \cong \mathcal{X}_p(A/F)$ as G -representations.

Proof. By Milne [13, p. 178 (a)], we have $W[p^n] = \text{Ind}_K^F A[p^n]$, and the isomorphism given by Shapiro’s lemma,

$$H^1(K, \text{Ind}_K^F A[p^n]) \longrightarrow H^1(F, A[p^n]),$$

descends to an isomorphism of Selmer groups. It is easy to check that it is compatible with the G -action.

This completes the proof of Theorem 2.1.

COROLLARY 2.5. *Let A/K be an abelian variety, and suppose F/K is Galois of odd degree. Then $\text{rk}_p(A/F) \equiv \text{rk}_p(A/K) \pmod{2}$.*

Proof. Since $G = \text{Gal}(F/K)$ has odd degree, its only self-dual irreducible representation is the trivial one. (Their number coincides with the number of self-inverse conjugacy classes of G , but these have odd order and, except for the trivial class, have no self-inverse elements.) So $\mathcal{X}_p(A/F) \cong \mathcal{X}_p(A/K) \oplus (\text{even-dimensional representation})$.

Remark 2.6. The p -parity conjecture for abelian varieties asserts that $(-1)^{\dim \mathcal{X}_p(A/K)}$ coincides with the root number $w(A/K)$, the (conjectural) sign in the functional equation for $L(A/K, s)$. If F/K is Galois of odd degree, then $w(A/F) = w(A/K)$ (see [17, 3.4.7, 4.2.4]), so the p -parity conjecture holds for A/F if and only if it holds for A/K , by the corollary.

COROLLARY 2.7. *Let A/K be an abelian variety, and suppose F/K is abelian. Let $K(\sqrt{d_i})$ be the quadratic extensions of K in F and write A_i for the corresponding quadratic twists of A . Then*

$$\text{rk}_p(A/F) \equiv \text{rk}_p(A/K) + \sum_i \text{rk}_p(A_i/K) \pmod{2}.$$

Proof. The self-dual irreducible complex representations of $\text{Gal}(F/K)$ are characters of order 1 or 2.

THEOREM 2.8 (Parity Conjecture in abelian extensions). *Let E/\mathbb{Q} be an elliptic curve. For every abelian extension F/\mathbb{Q} and every prime p ,*

$$\text{rk}_p(E/F) \equiv \text{ord}_{s=1} L(E/F, s) \pmod{2}.$$

Proof. With the notation from Corollary 2.7,

$$\text{rk}_p(E/F) \equiv \text{rk}_p(E/K) + \sum_i \text{rk}_p(E_i/K) \pmod{2}.$$

In view of the Parity Conjecture for elliptic curves over \mathbb{Q} ([3, Theorem 1.4]), the right-hand side agrees with the corresponding analytic ranks. By the functional equation, $\text{ord}_{s=1} L(E, \tau, s) = \text{ord}_{s=1} L(E, \tau^*, s)$ for every character τ of $\text{Gal}(F/K)$, so

$$\text{ord}_{s=1} L(E/F, s) \equiv \text{ord}_{s=1} L(E/\mathbb{Q}, s) + \sum_i \text{ord}_{s=1} L(E_i/\mathbb{Q}, s) \pmod{2}.$$

3. Regulator constants for Selmer groups

The central result of this section is:

THEOREM 3.1 (= Theorem 1.5). *Let $G = \text{Gal}(F/K)$ and let p be a prime number. Suppose A/K is principally polarised, and if $p=2$, assume furthermore that the principal polarisation on A is induced by a K -rational divisor. If $H_i, H'_j < G$ satisfy $\bigoplus \text{Ind}_{H_i}^G \mathbf{1} \cong \bigoplus \text{Ind}_{H'_j}^G \mathbf{1}$, then for every non-degenerate G -invariant \mathbb{Q}_p -bilinear pairing \langle, \rangle on $\mathcal{X} = \mathcal{X}_p(A/F)$,*

$$\text{ord}_p \frac{\prod_i \det\left(\frac{1}{|H_i|} \langle, \rangle \mid \mathcal{X}^{H_i}\right)}{\prod_j \det\left(\frac{1}{|H'_j|} \langle, \rangle \mid \mathcal{X}^{H'_j}\right)} \equiv \text{ord}_p \frac{\prod_i C(A/F^{H_i})}{\prod_j C(A/F^{H'_j})} \pmod{2}.$$

Proof. Write $S_1 = \coprod_i G/H_i$ and $S_2 = \coprod_j G/H'_j$. Since $\mathbb{Q}[S_1] \cong \mathbb{Q}[S_2]$, there is a G -injection $\phi: \mathbb{Z}[S_1] \rightarrow \mathbb{Z}[S_2]$ with finite cokernel, and it induces maps on abelian varieties

$$f_\phi: A \otimes \mathbb{Z}[S_1] \longrightarrow A \otimes \mathbb{Z}[S_2], \quad f_{\phi^t}: A \otimes \mathbb{Z}[S_2] \longrightarrow A \otimes \mathbb{Z}[S_1].$$

Applying Theorem 2.2 modulo rational squares (see also [3, corollary 4.5]),

$$\text{ord}_p \frac{\prod_i C(A/F^{H_i})}{\prod_j C(A/F^{H'_j})} \equiv \text{ord}_p Q(f_{\phi\phi^t}) \pmod{2}.$$

It remains to justify the last two steps in the following chain of equalities:

$$\begin{aligned} \text{ord}_p Q(f_{\phi\phi^t}) &= \text{ord}_p \text{coker}(f_{\phi\phi^t} \mid X_p(A \otimes \mathbb{Z}[S_2])) \\ &= \text{ord}_p \det(f_{\phi\phi^t} \mid \mathcal{X}_p(A \otimes \mathbb{Z}[S_2])) \\ &\stackrel{\text{Cor.3.4}}{\equiv} \text{ord}_p \det((\phi\phi^t)^* \mid \text{Hom}_G(\mathbb{Z}[S_2], \mathcal{X})) \\ &\stackrel{\text{Lem.3.2}}{\equiv} \text{ord}_p \frac{\prod_i \det\left(\frac{1}{|H_i|} \langle, \rangle \mid \mathcal{X}^{H_i}\right)}{\prod_j \det\left(\frac{1}{|H'_j|} \langle, \rangle \mid \mathcal{X}^{H'_j}\right)} \pmod{2}. \end{aligned}$$

LEMMA 3.2. *Suppose S_1, S_2 are finite G -sets, and $\phi: \mathbb{Z}[S_1] \rightarrow \mathbb{Z}[S_2]$ is a G -injection with finite cokernel. Write $\phi^t: \mathbb{Z}[S_2] \rightarrow \mathbb{Z}[S_1]$ for its transpose in the standard basis. For a $\mathbb{Q}_p G$ -representation V , write*

$$\phi^*: \text{Maps}_G(S_2, V) \longrightarrow \text{Maps}_G(S_1, V)$$

for the pullback of maps, and similarly for ϕ^t and $\phi\phi^t$.

If V has a G -invariant \mathbb{Q}_p -bilinear non-degenerate pairing \langle, \rangle , and we write $S_1 = \coprod G/H_i, S_2 = \coprod G/H'_j$ with $H_i, H'_j < G$, then

$$\text{ord}_p \det((\phi\phi^t)^*) \equiv \text{ord}_p \frac{\prod_i \det\left(\frac{1}{|H_i|} \langle, \rangle \mid V^{H_i}\right)}{\prod_j \det\left(\frac{1}{|H'_j|} \langle, \rangle \mid V^{H'_j}\right)} \pmod{2}.$$

Proof. We may identify

$$V^H = \text{Maps}_G(G/H, V) = \{f: G/H \rightarrow V \mid f(g \cdot s) = g \cdot f(s)\},$$

where the map from left to right is given by $f \mapsto f(1)$.

If S is a finite G -set, define an inner product on functions $S \rightarrow V$ by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{s \in S} \langle f_1(s), f_2(s) \rangle.$$

For $S = G/H$ it agrees with the inner product $\frac{1}{|H|} \langle \cdot, \cdot \rangle$ on V^H via the identification above. Indeed, if $f_1(1) = v_1$ and $f_2(1) = v_2$ then

$$\begin{aligned} (f_1, f_2) &= \frac{1}{|G|} \sum_{s \in G/H} \langle f_1(s), f_2(s) \rangle = \frac{1}{|G|} \sum_{s \in G/H} \langle s \cdot f_1(1), s \cdot f_2(1) \rangle \\ &= \frac{1}{|G|} \sum_{s \in G/H} \langle v_1, v_2 \rangle = \frac{1}{|H|} \langle v_1, v_2 \rangle. \end{aligned}$$

Therefore for $S_1 = \coprod G/H_i$ (and similarly for S_2),

$$\text{ord}_p \prod_i \det \left(\frac{1}{|H_i|} \langle \cdot, \cdot \rangle \mid V^{H_i} \right) \equiv \text{ord}_p \det((\cdot, \cdot) \mid \text{Maps}_G(S, V)) \pmod{2}.$$

An elementary computation shows that ϕ^* and $(\phi^t)^*$ are adjoint with respect to (\cdot, \cdot) , so picking a basis $\{e_k\}$ of $\text{Maps}_G(S_2, V)$,

$$\begin{aligned} \text{ord}_p \det((\cdot, \cdot) \mid \text{Maps}_G(S_1, V)) &\equiv \text{ord}_p \det((\phi^* e_k, \phi^* e_l)_{k,l}) \\ &\equiv \text{ord}_p \det((e_k, (\phi^t)^* \phi^* e_l)_{k,l}) \\ &\equiv \text{ord}_p \det((\phi \phi^t)^*) \det((\cdot, \cdot) \mid \text{Maps}_G(S_2, V)) \pmod{2}. \end{aligned}$$

LEMMA 3.3. *Let $A/K, F/K$ and G be as in Theorem 3.1, and suppose $\phi: \mathbb{Z}[S_1] \rightarrow \mathbb{Z}[S_2]$ is an injection of G -permutation modules with finite cokernel. There are natural vertical maps with finite kernels and cokernels that make the diagram*

$$\begin{array}{ccc} \text{Hom}_G(\mathbb{Z}[S_1], \text{Sel}_{p^\infty}(A/F)) & \xrightarrow{\phi^*} & \text{Hom}_G(\mathbb{Z}[S_2], \text{Sel}_{p^\infty}(A/F)) \\ \uparrow & & \uparrow \\ \text{Sel}_{p^\infty}(A \otimes \mathbb{Z}[S_1]/K) & \xrightarrow{f_\phi} & \text{Sel}_{p^\infty}(A \otimes \mathbb{Z}[S_2]/K) \end{array}$$

commute. Here ϕ^ is the pullback of maps induced by ϕ .*

Proof. Writing $S_1 = \coprod G/H_i, S_2 = \coprod G/H'_j$, we have a commutative diagram

$$\begin{array}{ccc} \text{Hom}_G(\prod \mathbb{Z}[G/H_i], H^1(F, A[p^n])) & \xrightarrow{\phi^*} & \text{Hom}_G(\prod \mathbb{Z}[G/H'_j], H^1(F, A[p^n])) \\ \text{eval}(1, 1, \dots, 1) \downarrow \cong & & \text{eval}(1, 1, \dots, 1) \downarrow \cong \\ \prod H^1(F, A[p^n])^{H_i} & & \prod H^1(F, A[p^n])^{H'_j} \\ \text{Res} \uparrow & & \text{Res} \uparrow \\ \prod H^1(F^{H_i}, A[p^n]) & & \prod H^1(F^{H'_j}, A[p^n]) \\ \text{eval}(1, 1, \dots, 1) \uparrow \cong & & \text{eval}(1, 1, \dots, 1) \uparrow \cong \\ H^1(K, \text{Hom}(\prod \mathbb{Z}[G/H_i], A[p^n])) & \xrightarrow{\phi^*} & H^1(K, \text{Hom}(\prod \mathbb{Z}[G/H'_j], A[p^n])) \end{array}$$

The restriction maps Res have bounded kernels and cokernels with respect to n . Taking the limit (and using a similar local diagram) proves the lemma.

COROLLARY 3.4. *In the situation of Lemma 3.3, there is a commutative diagram*

$$\begin{CD} \mathrm{Hom}_G(\mathbb{Z}[S_1], \mathcal{X}_p(A/F)) @<(\phi^t)^*<< \mathrm{Hom}_G(\mathbb{Z}[S_2], \mathcal{X}_p(A/F)) \\ @V \cong VV @VV \cong V \\ \mathcal{X}_p(A \otimes \mathbb{Z}[S_1]/K) @<f_\phi<< \mathcal{X}_p(A \otimes \mathbb{Z}[S_2]/K). \end{CD}$$

Proof. Apply Pontryagin duals $\mathrm{Hom}_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$ to the diagram of Lemma 3.3, and tensor with \mathbb{Q}_p . The fact that the Pontryagin duals in the top row are what the corollary asserts they are is general nonsense: for rings R, S and modules $\mathcal{A}_R, {}_R\mathcal{B}_S, \mathcal{C}_S$, there is a natural isomorphism (see e.g. [10, Theorem V.3.1, p. 144])

$$\eta: \mathrm{Hom}_S(\mathcal{A} \otimes_R \mathcal{B}, \mathcal{C}) \cong \mathrm{Hom}_R(\mathcal{A}, \mathrm{Hom}_S(\mathcal{B}, \mathcal{C}))$$

of abelian groups, defined for $h: \mathcal{A} \otimes_R \mathcal{B} \rightarrow \mathcal{C}$ by $[(\eta h)a](b) = h(a \otimes b)$. Apply this with $R = \mathbb{Z}[G], S = \mathbb{Z}_p, \mathcal{A} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S_i], \mathbb{Z}), \mathcal{B} = \mathrm{Sel}_{p^\infty}(\mathcal{A} \otimes \mathbb{Z}[S_i]/K)$ and $\mathcal{C} = \mathbb{Q}_p/\mathbb{Z}_p$. Note that $\phi: \mathbb{Z}[S_1] \rightarrow \mathbb{Z}[S_2]$ induces the transpose map

$$\phi^t: \mathbb{Z}[S_2] = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S_2], \mathbb{Z}) \rightarrow \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[S_1], \mathbb{Z}) = \mathbb{Z}[S_1].$$

We illustrate the applications of Theorem 1.6 with a few examples. As in the theorem, suppose A/K is a principally polarised abelian variety, F/K a Galois extension, and p a fixed prime. Write m_τ for the multiplicity of τ in $\mathcal{X}_p(A/F)$, and $\Theta = \sum H_i - \sum H'_j$ for a relation $\Theta: \bigoplus_i \mathrm{Ind}_{H_i}^G \mathbf{1} \cong \bigoplus_j \mathrm{Ind}_{H'_j}^G \mathbf{1}$.

Example 3.5. Suppose p is an odd prime and $G = \mathrm{Gal}(F/K) \cong D_{2p}$ is dihedral. Let M, L be intermediate fields of degree 2 and p over K , respectively. The group G has three \mathbb{Q}_p -irreducible representations, which are all self-dual: trivial $\mathbf{1}$, sign ϵ and $(p-1)$ -dimensional ρ . There is a unique (up to multiples) relation between permutation representations,

$$\Theta = \{1\} - 2 \mathrm{Gal}(F/L) - \mathrm{Gal}(F/M) + 2 G.$$

A simple computation shows that $\mathcal{C}(\Theta, \mathbf{1}) = \mathcal{C}(\Theta, \epsilon) = \mathcal{C}(\Theta, \rho) = p$, so $\mathcal{S}_\Theta = \{\mathbf{1}, \epsilon, \rho\}$. By Theorem 1.6,

$$m_{\mathbf{1}} + m_\epsilon + m_\rho \equiv \mathrm{ord}_p \frac{C(A/F)C(A/K)^2}{C(A/M)C(A/L)^2} \equiv \mathrm{ord}_p \frac{C(A/F)}{C(A/M)} \pmod{2}.$$

Example 3.6. Suppose p is an odd prime and $G = \begin{pmatrix} 1 & * \\ 0 & \ast \end{pmatrix} \subset \mathrm{GL}_2(\mathbb{F}_p)$. Let M, L be intermediate fields of degree $p-1$ and p over K , respectively. The group G has $p-1$ one-dimensional representations, all of which are realisable over \mathbb{Q}_p and factor through $\mathrm{Gal}(M/K)$; write ϵ for the one of order 2. The only other irreducible representation ρ of G has dimension $p-1$ and can be realised over \mathbb{Q} . There is a relation between permutation representations,

$$\Theta = \{1\} - (p-1) \mathrm{Gal}(F/L) - \mathrm{Gal}(F/M) + (p-1) G.$$

Here $\mathcal{S}_\Theta = \{\mathbf{1}, \epsilon, \rho\}$, and Theorem 1.6 implies

$$m_{\mathbf{1}} + m_\epsilon + m_\rho \equiv \mathrm{ord}_p \frac{C(A/F)C(A/K)^{p-1}}{C(A/M)C(A/L)^{p-1}} \equiv \mathrm{ord}_p \frac{C(A/F)}{C(A/M)} \pmod{2}.$$

Example 3.7. Suppose p is an odd prime, and consider

$$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, \quad U_1 = \begin{pmatrix} \square & * \\ 0 & * \end{pmatrix}, \quad U_2 = \begin{pmatrix} * & * \\ 0 & \square \end{pmatrix} < G = \mathrm{GL}_2(\mathbb{F}_p),$$

where \square stands for non-zero squares. An elementary computation with double cosets shows that

$$\begin{aligned} B \backslash G / U_i &= B \backslash G / B = \{B, G - B\} \\ U_i \backslash G / U_i &= \{U_i, B - U_i, G - B\} \\ U_1 \backslash G / U_2 &= \{B, \Sigma, \Sigma'\} \end{aligned}$$

with Σ and Σ' the sets of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $c(bc - ad)$ non-zero square and non-square, respectively. Since $(\mathrm{Ind}_H^G \mathbf{1}, \mathrm{Ind}_{H'}^G \mathbf{1}) = |H \backslash G / H'|$,

$$\mathrm{Ind}_B^G \mathbf{1} = \mathbf{1} \oplus \sigma, \quad \mathrm{Ind}_{U_1}^G \mathbf{1} = \mathrm{Ind}_{U_2}^G \mathbf{1} = \mathbf{1} \oplus \sigma \oplus \rho,$$

where σ (Steinberg) is irreducible of dimension p , and ρ irreducible of dimension $p + 1$. In particular,

$$\Theta = U_1 - U_2$$

is a relation between permutation representations. We will compute the regulator constants $\mathcal{C}(\Theta, \tau)$ for this relation.

If τ is not $\mathbf{1}$, σ or ρ , then by Frobenius reciprocity

$$\dim \tau^{U_i} = \langle \mathbf{1}_{U_i}, \mathrm{Res}_{U_i} \tau \rangle = \langle \mathbf{1} + \sigma + \rho, \tau \rangle = 0,$$

so $\mathcal{C}(\Theta, \tau) = 1$. On the other hand,

$$\mathcal{C}(\Theta, \mathbf{1}) = 1, \quad \mathcal{C}(\Theta, \sigma) = 1, \quad \mathcal{C}(\Theta, \rho) = p.$$

To see this, it suffices to verify that $\mathcal{C}(\Theta, \mathbf{1}) = \mathcal{C}(\Theta, \mathrm{Ind}_B^G \mathbf{1}) = 1$ and $\mathcal{C}(\Theta, \mathrm{Ind}_{U_1}^G \mathbf{1}) = p$. These are permutation representations, so they come with the standard “identity” pairing \langle, \rangle . It satisfies

$$\det \left(\frac{1}{|N|} \langle, \rangle |(\mathrm{Ind}_H^G \mathbf{1})^N \right) = \prod_{x \in N \backslash G / H} \frac{|NxH|}{|N||H|},$$

which is easy to compute from the explicit description of double cosets.

By Theorem 1.6, for any principally polarised abelian variety A/K ,

$$\mathrm{rk}_p(A/F^{U_1}) - \mathrm{rk}_p(A/F^B) = m_\rho \equiv \mathrm{ord}_p \frac{C(A/F^{U_1})}{C(A/F^{U_2})} \pmod{2}.$$

Note that although F^{U_1} and F^{U_2} are arithmetically equivalent fields (they have the same zeta-function), the right-hand side need not be 0. For instance, take an elliptic curve E/\mathbb{Q} of prime conductor $l \neq p$ and split multiplicative reduction at l , $p \nmid \mathrm{ord}_l j(E)$, l a primitive root modulo p , and $\mathrm{Gal}(\mathbb{Q}(E[p])/\mathbb{Q}) = \mathrm{GL}_2(\mathbb{F}_p) = G$. (E.g. take $E = X_1(11)$, $p = 3$). Then the decomposition and inertia subgroups of l in G are

$$D = \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad I = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

It is easy to see that l decomposes in F^{U_1} and F^{U_2} as

$$l = \mathfrak{p}_1 \mathfrak{p}_2^p \mathfrak{p}_3^p \quad \text{and} \quad l = \mathfrak{q}_1 \mathfrak{q}_2^p \mathfrak{q}_3$$

with p_1 and q_2 of residue degree 2 and the rest of residue degree 1 over l . So

$$C(E/F^{U_1}) = p^2 \cdot c_l(E/\mathbb{Q})^3, \quad C(E/F^{U_2}) = p \cdot c_l(E/\mathbb{Q})^3,$$

and m_ρ is therefore odd.

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