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The equivariant cohomology ring of weighted projective space

BY ANTHONY BAHRI

Department of Mathematics, Rider University, Lawrenceville, NJ 08648, *U.S.A.*
e-mail: bahri@rider.edu

MATTHIAS FRANZ

*Department of Mathematics, University of Western Ontario, London, ON N*6*A* 5*B*7, *Canada*
e-mail: mfranz@uwo.ca

AND NIGEL RAY

*School of Mathematics, University of Manchester, Manchester M*13 9*PL.*
e-mail: nigel.ray@manchester.ac.uk

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Abstract

We describe the integral equivariant cohomology ring of a weighted projective space in terms of piecewise polynomials, and thence by generators and relations. We deduce that the ring is a perfect invariant, and prove a Chern class formula for weighted projective bundles.

1. *Introduction*

Let $\chi = (\chi_0, \ldots, \chi_n)$ be a vector of positive natural numbers. The associated *weighted projective space* is the quotient

$$\mathbb{P}(\chi) = S^{2n+1}/S^1\langle\chi_0, \ldots, \chi_n\rangle, \tag{1·1}$$

where the numbers χ_i indicate the weights with which S^1 acts on the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$, by

$$g \cdot (x_0, \ldots, x_n) = \big(g^{\chi_0}x_0, \ldots, g^{\chi_n}x_n\big). \tag{1·2}$$

So $\mathbb{P}(1, \ldots, 1)$ is the standard projective space $\mathbb{CP}^n$. Note that $\mathbb{P}(\chi)$ is equipped with an action of the n-dimensional torus

$$T = (S^1)^{n+1}/S^1\langle\chi_0, \ldots, \chi_n\rangle, \tag{1·3}$$

where the quotient is defined by analogy with (1·1). We give an explicit example of such an action in (4·1), for $n = 3$. Different weight vectors may also give equivalent weighted projective spaces; we will elaborate on this aspect in Section 5.

Kawasaki [**14**] has computed the ordinary cohomology ring of $\mathbb{P}(\chi)$ with integer coefficients. Additively, the cohomology is isomorphic to that of $\mathbb{CP}^n$, but the multiplication is different. More precisely, if c_1 is a generator of the group $H^2(\mathbb{P}(\chi))$, then $H^*(\mathbb{P}(\chi))$ is generated as a ring by elements c_m, where $1 \leqslant m \leqslant n$ and

$$c_1^m = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)^m}{\operatorname{lcm}\left\{\prod_{i \in I} \chi_i : |I| = m\right\}} c_m \tag{1·4}$$

in $H^{2m}(\mathbb{P}(\chi))$. The multiplication is induced accordingly.

In this paper we study $H_T^*(\mathbb{P}(\chi))$, the T-equivariant cohomology of $\mathbb{P}(\chi)$ with integer coefficients; it is defined as the ordinary cohomology $H^*(\mathbb{P}(\chi)_T)$ of the Borel construction $\mathbb{P}(\chi)_T = ET \times_T \mathbb{P}(\chi)$. Our main result, Theorem 3·7, describes $H_T^*(\mathbb{P}(\chi))$ in terms of generators and relations.

We give two applications. Firstly, we show how to recover the weight vector χ from $H_T^*(\mathbb{P}(\chi))$, thereby establishing that different weighted projective spaces have different integral equivariant cohomology rings. Secondly, we consider weighted projective bundles. For any direct sum $D = L_1 \oplus \cdots \oplus L_n$ of complex line bundles over a base space X, the cohomology ring $H^*(\mathbb{P}(D))$ of the projectivisation is a module over $H^*(X)$. Its algebra structure is determined by the single relation

$$\prod_{i=0}^{n} (\xi + c_1(L_i)) = 0, \tag{1·5}$$

where $c_1(L_i) \in H^2(X)$ and $-\xi \in H^2(\mathbb{P}(D))$ denote the Chern classes of L_i and of the canonical complex line bundle respectively. Al Amrani [**1**] has stated a generalisation of (1·5) to weighted projective bundles and proved it in a special case. Theorem 6·2 establishes his relation in general.

We consider our calculations as lying in the realm of toric topology, and will elaborate on this theme in a subsequent document. Readers who require background information on equivariant topology may consult [**2**], or the survey articles in [**15**].

2. *From equivariant cohomology to piecewise polynomials*

By a *ring* we always mean a graded commutative ring with unit element. All rings we consider happen to be concentrated in even degrees, so that they are commutative in the ordinary sense.

Remark 1. There are several ways to describe the divisibility of the powers c_1^m in $H^*(\mathbb{P}(\chi))$. Kawasaki looks at the p-contents of the weights for each prime p separately. If $q_0(p), \ldots, q_n(p)$ are their p-contents, in increasing order, then

$$c_1^m = \prod_p \frac{q_n(p)^m}{q_n(p) \cdots q_{n-m+1}(p)} c_m. \tag{2·1}$$

Kawasaki also considers sets I of size $m+1$ instead of m and writes

$$c_1^m = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)^m}{\operatorname{lcm}\left\{\gcd\{\chi_i : i \in I\}^{-1} \prod_{i \in I} \chi_i : |I| = m+1\right\}} c_m \tag{2·2}$$

[**14**, p. 248], as does Al Amrani [**1**, section I·5]. Taking products of $m+1$ weights and dividing by their greatest common divisor, as in the denominator of (2·2), removes the smallest p-content for each prime p. Computing the least common multiple over all such terms then gives the product of the m largest p-contents of all weights, in accordance with the denominators of (1·4) and (2·1).

Now let $\iota \colon \mathbb{P}(\chi) \to \mathbb{P}(\chi)_T$ be the inclusion of a fibre into the Borel construction.

LEMMA 2·1. *As an $H^*(BT)$-module, $H_T^*(\mathbb{P}(\chi))$ is free of rank $n+1$: as a ring, it is generated by the image of $H^2(BT)$ in $H_T^2(\mathbb{P}(\chi))$, together with any subgroup $A^* < H_T^*(\mathbb{P}(\chi))$ that surjects onto $H^{>0}(\mathbb{P}(\chi))$ under ι^*.*

Proof. By Kawasaki, $H^*(\mathbb{P}(\chi))$ is free over $\mathbb{Z}$ and concentrated in even degrees. So the Serre spectral sequence of the fibration $\mathbb{P}(\chi) \to \mathbb{P}(\chi)_T \to BT$ degenerates at the E_2 level, and $H_T^*(\mathbb{P}(\chi))$ is isomorphic as $H^*(BT)$-modules to $H^*(\mathbb{P}(\chi)) \otimes H^*(BT)$ by the Leray–Hirsch theorem. The isomorphism is induced by any additive section to ι^*, which we may assume takes values in A^*. This proves the claim.

The equivariant cohomology of $\mathbb{CP}^n$ is well known, and may be described conveniently in the context of toric varieties. Indeed, $\mathbb{P}(\chi)$ is an n-dimensional projective toric variety for every χ. It may be constructed from any complete simplicial fan Σ whose rays pass through vectors $v_0, \ldots, v_n$ that span an n–dimensional lattice N, and satisfy the relation

$$\chi_0 v_0 + \cdots + \chi_n v_n = 0. \tag{2·3}$$

Every such Σ is the normal fan of a lattice n-simplex in $N \otimes_{\mathbb{Z}} \mathbb{R}$, and hence polytopal; we refer to Fulton [**10**, section 2·2], or the nice overview in [**12**, section 4·1], for a full discussion.

In particular, $H_T^*(\mathbb{CP}^n)$ is isomorphic to the integral Stanley–Reisner algebra

$$\mathbb{Z}[a_0, \ldots, a_n]/(a_0 \cdots a_n) \tag{2·4}$$

of the appropriate Σ, where each generator a_i corresponds to v_i, and has degree 2. In other words, the only relation amongst the generators is

$$\prod_{i=0}^{n} a_i = 0. \tag{2·5}$$

The situation for singular varieties is less straightforward, and the $\mathbb{P}(\chi)$ offer a natural family of test cases. Our aim is to generalise (2·4), and express $H_T^*(\mathbb{P}(\chi))$ in terms of generators and relations for arbitrary χ. We use the language of piecewise polynomials, to which we now turn.

A function $f\colon N \to \mathbb{Z}$ is called *piecewise polynomial* on Σ if it coincides with some globally defined polynomial $g \in \mathbb{Z}[N]$ on each cone σ. Such functions are closed under pointwise addition and multiplication, and form an algebra $PP[\Sigma]$ over the ring of global polynomials $\mathbb{Z}[N]$. We grade both $PP[\Sigma]$ and $\mathbb{Z}[N]$ by twice the degree of homogeneous elements, and use the Chern classes of the n canonical line bundles to identify $\mathbb{Z}[N]$ with $H^*(BT)$.

PROPOSITION 2·2. *Let Σ be a polytopal fan in N, and X_Σ the associated compact projective toric variety: if $H^*(X_\Sigma)$ is concentrated in even degrees, then $H_T^*(X_\Sigma)$ is isomorphic to $PP[\Sigma]$ as an algebra over $H^*(BT)$.*

Proof. Set $X = X_\Sigma$, and denote the orbit space by X/T. The latter may be identified with a convex polytope P_Σ, whose normal fan is Σ. Following Goresky and MacPherson [**11**], we deduce that X is homeomorphic to a quotient space of $T \times P_\Sigma$, and may therefore be expressed as a finite T-CW complex [**15**, p. 13] with connected isotropy groups. As in the proof of Lemma 2·1, the Serre spectral sequence for the fibration $X \to X_T \to BT$ degenerates at the E_2 level because $H^*(X)$ is concentrated in even degrees. Hence, by a result of Franz–Puppe [**9**], the Chang–Skjelbred sequence

$$0 \longrightarrow H_T^*(X) \xrightarrow{j^*} H_T^*(X^T) \xrightarrow{\delta} H_T^{*+1}(X_1, X^T) \tag{2·6}$$

is exact for integral coefficients. Here X^T denotes the T-fixed points, X_1 the union of X^T and all 1-dimensional orbits, j the inclusion $X^T \to X$ and δ the differential of the long exact cohomology sequence for the pair (X_1, X^T). We may identify the kernel of δ with the algebra $PP[\Sigma]$, as follows.

Write $\mathcal{O}_\sigma$ for the orbit under the complexification $T_{\mathbb{C}}$ of T corresponding to the cone $\sigma \in \Sigma$, and $\mathbb{Z}[\sigma]$ for the polynomials with integer coefficients on the linear hull of σ. Any such polynomial is uniquely determined by its restriction to σ.

When σ is n-dimensional, we have that

$$H_T^*(\mathcal{O}_\sigma) = H^*(BT) = \mathbb{Z}[\sigma]. \tag{2·7}$$

For $(n-1)$-dimensional τ, we denote the isotropy group of $\mathcal{O}_\tau$ by T_τ. Then the action of the circle T/T_τ on the closure $\bar{\mathcal{O}}_\tau$ is isomorphic to the standard action of S^1 on $\mathbb{CP}^1$, whose fixed points we write as 0 and ∞. We obtain

$$H_T^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) = H^*(BT_\tau) \otimes H_{T/T_\tau}^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \tag{2·8}$$

and

$$H_{T/T_\tau}^*(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \cong H_{S^1}^*(\mathbb{CP}^1, \{0, \infty\}) \cong \mathbb{Z}[+1]; \tag{2·9}$$

hence

$$H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \cong \mathbb{Z}[\tau]. \tag{2·10}$$

Moreover, whenever $\tau \subset \sigma$ is a facet, the differential

$$H_T^*(\mathcal{O}_\sigma) \longrightarrow H_T^{*+1}(\bar{\mathcal{O}}_\tau, \partial\mathcal{O}_\tau) \tag{2·11}$$

is the canonical restriction $\mathbb{Z}[\sigma] \to \mathbb{Z}[\tau]$, multiplied by ± 1 according to the orientation of the interval $\mathcal{O}_\tau/T \approx (0, \infty)$.

It follows that the differential δ of (2·6) is a signed sum of restrictions

$$\delta\colon \bigoplus_{\sigma\in\Sigma^n} \mathbb{Z}[\sigma] \longrightarrow \bigoplus_{\tau\in\Sigma^{n-1}} \mathbb{Z}[\tau] \tag{2·12}$$

of sums of polynomial algebras, whose component into $\mathbb{Z}[\tau]$ is the difference of the restrictions of the polynomials on the two n-dimensional cones having τ as their common facet. So the kernel consists of those collections of polynomials on n-dimensional cones which may be glued along their common facets. This corresponds to the requirement that the polynomials agree on *any* intersection $\tau = \sigma \bigcap \sigma'$, because σ and σ' are connected by a sequence of n-dimensional cones, each of which contains τ and shares a facet with the next. (In other words, Σ is a *hereditary fan* [**4**].) Thus δ has kernel $PP[\sigma]$, as required.

Finally, let f be the piecewise polynomial corresponding to a class $\alpha \in H_T^*(X)$. By construction, the polynomial coinciding with f on an n-dimensional cone σ is the image of α under the restriction

$$H_T^*(X) \to H_T^*(\mathcal{O}_\sigma) = H^*(BT) = \mathbb{Z}[N]. \tag{2·13}$$

Since the composition $H^*(BT) \to H_T^*(X) \to H_T^*(\mathcal{O}_\sigma) = H^*(BT)$ is the identity, it follows that the map $H^*(BT) \to H_T^*(X)$ corresponds to the inclusion of the algebra of global polynomials.

Remark 2. Proposition 2·2 is actually valid for arbitrary fans, but we have been unable to locate a reference identifying X_Σ as a T-CW complex in general; we are therefore preparing an independent proof of this fact [**8**]. A more circuitous alternative may be developed by following suggestions of [**9**, remark 1·4].

Payne [**16**] has shown that $PP[\Sigma]$ is also isomorphic to the equivariant Chow ring of X_Σ for any fan Σ. Further developments are documented in [**13**].

The integral equivariant cohomology of any smooth, not necessarily compact toric variety X_Σ is given by the Stanley–Reisner algebra of Σ [**3**], [**6**]; or equivalently, by $PP[\Sigma]$ [**5**]. A canonical isomorphism between the two is defined by assigning the Courant function a_ρ of the ray ρ to the Stanley–Reisner generator corresponding to ρ. The function a_ρ is piecewise linear on Σ, and assumes the value 1 on the generator of ρ and 0 on all other rays. It is well-defined because the smoothness of X_Σ implies that the rays of any cone may be completed to a basis of the lattice N. Brion was probably the first to note the relationship between piecewise polynomials and the Chang–Skjelbred sequence.

Similarly, when Σ is simplicial the rational equivariant cohomology $H_T^*(X_\Sigma; \mathbb{Q})$ is given by the rational Stanley–Reisner algebra of Σ; or equivalently, by $PP[\Sigma] \otimes \mathbb{Q}$ [**10**, p. 107]. In particular, there is an isomorphism

$$H_T^*(\mathbb{P}(\chi); \mathbb{Q}) \cong \mathbb{Q}[a_0, \ldots, a_n]/(a_0 \ldots a_n). \qquad (2\cdot 14)$$

3. *Generators of the ring of piecewise polynomials*

For $i = 0, \ldots, n$ we will write $\sigma_i \in \Sigma$ for the full-dimensional cone spanned by all rays except v_i. This cone is simplicial and full-dimensional because the set $\{v_j : j \neq i\}$ is a basis of the $\mathbb{Q}$-vector space $N \otimes_{\mathbb{Z}} \mathbb{Q}$ for each i. Moreover, given a piecewise polynomial f, we will denote the unique polynomial which coincides with f on σ_i by $f^{(i)}$. We call a piecewise polynomial *reduced* if it is not divisible in $PP[\Sigma]$ by any rational prime.

Let b_{ij}, where $i \neq j$, be the reduced linear function that assumes a positive value on v_i and vanishes on all v_k, for $i \neq k \neq j$.

LEMMA 3·1. *We have that*

$$b_{ij}(v_i) = \frac{\chi_j}{\gcd(\chi_i, \chi_j)} = \frac{\operatorname{lcm}(\chi_i, \chi_j)}{\chi_i}. \qquad (3\cdot 1)$$

Proof. Applying b_{ij} to the relation (2·3) yields

$$\chi_i\, b_{ij}(v_i) = -\chi_j\, b_{ij}(v_j). \qquad (3\cdot 2)$$

Since b_{ij} is reduced and v_i and v_j span $N/\ker b_{ij} \cong \mathbb{Z}$, the values $b_{ij}(v_i)$ and $b_{ij}(v_j)$ must be coprime. This implies the claimed formula.

PROPOSITION 3·2. *The b_{ij} generate $N^\vee$, the lattice dual to N.*

Proof. Multiplying all weights by a constant factor changes neither the fan Σ nor the functions b_{ij}, so we may assume that the greatest common divisor of the weights equals 1. For given j, let N_j be the span of the linearly independent set $V_j = \{v_i : i \neq j\}$ and $N_j^\vee$ its dual. By Lemma 3·1, the restriction of each b_{ij} with $i \neq j$ to N_j is divisible by a divisor of χ_j, and the quotient is an element of the basis dual to V_j.

Let $M_j < N^\vee$ denote the sublattice generated by those b_{ij} for which $i \neq j$. Our goal is then to show that $M = N^\vee$, where M is generated by the M_j.

We have that

$$N_j^\vee/N^\vee = (N_j^\vee/M_j)/(N^\vee/M_j). \tag{3·3}$$

Therefore, the order of $N^\vee/M_j$ divides that of $N_j^\vee/M_j$, which itself divides χ_j^n from above. Hence, the order of $N^\vee/M_j$ also divides χ_j^n, and the same applies to $N^\vee/M$ because

$$N^\vee/M = (N^\vee/M_j)/(M/M_j). \tag{3·4}$$

This implies that the order of $N^\vee/M$ divides the greatest common divisor of all χ_j^n, which we assumed to be 1.

Let a_i denote the *Courant function* corresponding to v_i, for $0 \leqslant i \leqslant n$. By this we mean the reduced piecewise linear function that assumes a positive value on v_i and vanishes on all v_j for $j \neq i$. Each σ_j is simplicial, so a_i is well-defined.

LEMMA 3·3. *Together with the linear functions, each a_i generates the piecewise linear functions in $PP[\Sigma]$.*

Proof. Let f be piecewise linear. Then $f - f^{(i)}$ vanishes on σ_i, and is therefore a multiple of a_i.

LEMMA 3·4. *We have that*

$$a_i(v_i) = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_i} \qquad \text{and} \qquad a_i^{(j)} = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}(\chi_i, \chi_j)}\, b_{ij} \tag{3·5}$$

in $PP[\Sigma]$, for all $i \neq j$.

Proof. By Lemma 3·1 we can define a piecewise linear function f on Σ by setting $f^{(i)} = 0$, and $f^{(j)}$ equal to the given formula for $j \neq i$. Now let χ_k be a weight with maximal p-content, for some prime p. If $k = i$, then p cannot divide $f^{(j)}$ for any $j \neq k$; and if $k \neq i$, then p cannot divide $f^{(k)}$. So f is reduced, and $f = a_i$.

LEMMA 3·5. *In $PP[\Sigma]$, the relation*

$$b_{ij} = \frac{\operatorname{lcm}(\chi_i, \chi_j)}{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}\,(a_i - a_j) \tag{3·6}$$

holds for all $i \neq j$.

Proof. By Lemma 3·4, we have that

$$a_i^{(j)} = -a_j^{(i)} = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}(\chi_i, \chi_j)}\, b_{ij}, \tag{3·7}$$

and $a_i^{(i)} = -a_j^{(j)} = 0$; so $(a_i - a_j)^{(i)} = a_i^{(j)} = (a_i - a_j)^{(j)}$. Furthermore, by Lemma 3·3, $a_i - a_j$ may be written as $ma_k + r$ for some integer m and linear function r. But every a_k restricts to distinct linear functions on maximal cones σ_i and σ_j, so $m = 0$. Hence $a_i - a_j$ is linear, and divisible as claimed.

We now consider higher-degree analogues of the Courant functions a_i.

LEMMA 3·6. *For any nonempty subset $I \subset \{0, \ldots, n\}$, the function $\prod_{i\in I} a_i$ is divisible by*

$$\prod_{i\in I} \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}} \tag{3·8}$$

in $PP[\Sigma]$.

Proof. We look at each prime p separately. If the maximal p-content occurs in χ_j for some $j \notin I$, then there is nothing to prove because the p-content of (3·8) is 1. We can therefore assume that it occurs in χ_k for some $k \in I$.

Choose an $i \in I$ and denote the p-contents of χ_i and χ_k by q_i and q_k respectively. Then all $a_i^{(j)}$ with $j \notin I$ are divisible by q_k/q_i, which is greater than or equal to the p-content of

$$\frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}}. \tag{3·9}$$

Taking the product over all $i \in I$ finishes the proof.

Hence, for $I \subset \{0, \ldots, n\}$ we may define the piecewise polynomial

$$a_I = \left(\prod_{i \in I} \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}} \right)^{-1} \prod_{i \in I} a_i \tag{3·10}$$

in the $2|I|$-dimensional component of $PP[\Sigma]$.

THEOREM 3·7. *The ring $H_T^*(\mathbb{P}(\chi))$ is generated by the functions a_I and b_{ij}, where $1 \leqslant |I| \leqslant n$ and $i \nparallel j$ respectively. The only relations are* (2·5), (3·6) *and* (3·10).

Proof. From (2·14), $H_T^*(\mathbb{P}(\chi); \mathbb{Q})$ is generated by the a_i subject only to the relation (2·5). The relations (3·6) and (3·10) show that the a_I with $|I| > 1$ and the b_{ij} are redundant over $\mathbb{Q}$, so adding these generators and relations gives an isomorphic ring. Since there are no more relations between the a_I and b_{ij} in $H_T^*(\mathbb{P}(\chi); \mathbb{Q})$, the same is true in $H_T^*(\mathbb{P}(\chi))$; for the latter is free over $\mathbb{Z}$, and injects into $H_T^*(\mathbb{P}(\chi); \mathbb{Q})$. It remains to show that these elements are ring generators.

By Proposition 3·2, the b_{ij} generate the linear functions, which are the image of $H^2(BT)$ in $H_T^2(\mathbb{P}(\chi))$. Hence, by Lemma 2·1, it suffices to show that the subgroup generated by the a_I surjects onto $H^*(\mathbb{P}(\chi))$. In other words, we have to show that c_m lies in the span of $\{\iota^*(a_I) : |I| = m\}$ for each $1 \leqslant m \leqslant n$.

For $m = 1$, this is true by Lemma 3·3 because we know ι^* itself to be surjective. Moreover, Lemma 3·5 implies that all elements a_i are mapped to the same element of $H^2(\mathbb{P}(\chi))$. This must necessarily be a generator, which we may assume to be c_1.

For $1 < m \leqslant n$, we obtain

$$\iota^*(a_I) = \left(\prod_{i \in I} \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}} \right)^{-1} \prod_{i \in I} \iota^*(a_i) \tag{3·11 a}$$

$$= \left(\prod_{i \in I} \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}} \right)^{-1} c_1^m \tag{3·11 b}$$

$$= \frac{\prod_{i \in I} \operatorname{lcm}\{\chi_i, \chi_j : j \notin I\}}{\operatorname{lcm}\left\{ \prod_{j \in J} \chi_j : |J| = m \right\}} c_m \quad \text{by (1·4).} \tag{3·11 c}$$

We must show that these multiples of c_m generate $H^{2m}(\mathbb{P}(\chi))$. Once more, we consider each prime p separately, and let I be the set of indices (which need not be unique) that correspond to m weights with greatest p-content. Since this is also the set J which maximises the p-content of the denominator of (3·11 c), we conclude that for each p there appears a multiple of c_m whose p-content is 1. In other words, the greatest common divisor of all multiples is 1, as required.

4. *An example*

We illustrate the results of the preceding section in the case $\chi = (1, 2, 3, 4)$, which confirms that the elements b_{ij} cannot be omitted from the statement of Theorem 3·7.

We choose $v_0 = (-2, -3, -4)$, $v_1 = (1, 0, 0)$, $v_2 = (0, 1, 0)$, and $v_3 = (0, 0, 1)$. So isomorphisms $(S^1)^4/S^1\langle 1, 2, 3, 4\rangle \leftrightarrow (S^1)^3$ identifying the torus T of (1·3) are induced by $(t_0, t_1, t_2, t_3) \mapsto (t_0^{-2}t_1, t_0^{-3}t_2, t_0^{-4}t_3)$ and $(u_1, u_2, u_3) \mapsto (1, u_1, u_2, u_3)$ respectively, and its action on $\mathbb{P}(1, 2, 3, 4)$ is equivalent to that of $(S^1)^3$, given by

$$(u_1, u_2, u_3) \cdot [x_0, x_1, x_2, x_3] = [x_0, u_1x_1, u_2x_2, u_3x_3] \tag{4·1}$$

on homogeneous coordinates.

Writing an element f of $PP[\Sigma]$ as $f = (f^{(0)}, f^{(1)}, f^{(2)}, f^{(3)})$, and the canonical basis of $N^\vee$ as (x, y, z), we deduce

$$a_0 = (0, -6x, -4y, -3z), \tag{4·2a}$$
$$a_1 = (6x, 0, 6x - 4y, 6x - 3z), \tag{4·2b}$$
$$a_2 = (4y, -6x + 4y, 0, 4y - 3z), \tag{4·2c}$$
$$a_3 = (3z, -6x + 3z, -4y + 3z, 0). \tag{4·2d}$$

Each a_i is reduced, although individual components $a_i^{(j)}$ may have non-trivial divisors. The situation changes for products $\prod_{i\in I} a_i$, because components $a_i^{(j)}$ with $j \in I$ are multiplied by 0; this is the essence of Lemma 3·6. Since $\operatorname{lcm}\{1, 2, 3, 4\} = 12$, we obtain

$$a_{01} = a_0a_1, \quad a_{02} = a_0a_2/3, \quad a_{03} = a_0a_3/2, \tag{4·3a}$$
$$a_{12} = a_1a_2/3, \quad a_{13} = a_1a_3/2, \quad a_{23} = a_2a_3/6, \tag{4·3b}$$
$$a_{012} = a_0a_1a_2/9, \quad a_{013} = a_0a_1a_3/8, \tag{4·3c}$$
$$a_{023} = a_0a_2a_3/36, \quad a_{123} = a_1a_2a_3/72. \tag{4·3d}$$

Any globally linear function obtained from the a_i is a linear combination of

$$a_1 - a_0 = 6x, \quad a_2 - a_0 = 4y, \quad a_3 - a_0 = 3z, \tag{4·4a}$$
$$a_2 - a_1 = 4y - 6x, \quad a_3 - a_1 = 3z - 6x, \quad a_3 - a_2 = 3z - 4y, \tag{4·4b}$$

by Proposition 3·2 and Lemma 3·5. The functions $b_{10} = x$, $b_{20} = y$ and $b_{30} = z$ are obviously not in the span of the a_i. On the other hand, given x, y and z, the remaining b_{ij} are redundant, as are most of the a_I. In fact we may write

$$H_T^*(\mathbb{P}(1, 2, 3, 4)) \cong \mathbb{Z}[x, y, z, a_3, a_{23}, a_{123}]/(a_0a_1a_2a_3) \tag{4·5}$$

by Lemma 2·1, subject to the relations above. But we know of no canonical choice for a minimal set of generators.

For examples in which $\chi_i \nmid 1$ for any i, the construction of a fan Σ and an explicit T-action on $\mathbb{P}(\chi)$ is more difficult, and amounts to completing $\chi/\gcd(\chi_0, \ldots, \chi_n)$ to a lattice basis. Further details may be found in [**12**, section 4·2].

5. *Recovering the weights*

It is clear from the definition (1·1) that $\mathbb{P}(\chi)$ does not change if all weights are multiplied by the same factor. In particular, we may always divide the weights by their greatest common divisor, as in the proof of Proposition 3·2. Moreover, if every weight except χ_i is divisible by some prime p, then $\mathbb{P}(\chi)$ is equivariantly isomorphic to $\mathbb{P}(\chi')$, where

$\chi' = (\chi_0/p, \ldots, \chi_{i-1}/p,\ \chi_i,\ \chi_{i+1}/p, \ldots, \chi_n/p)$ [7]. This may be seen from the toric viewpoint: for (2·3) implies that v_i is also divisible by p, and continues to hold when v_i and χ are replaced by v_i/p and χ' respectively. So the fan Σ is unchanged, and the corresponding toric varieties are isomorphic.

By repeating these simplifications, we can always ensure that two or more weights are not divisible by p, for each prime p. The resulting weight vector is uniquely defined by χ (up to order), and the corresponding weights are called *normalised*.

THEOREM 5·1. *The graded ring $H_T^*(\mathbb{P}(\chi))$ determines the normalised weights.*

Proof. The length $n+1$ of χ is given by the rank of the free abelian group $H_T^2(\mathbb{P}(\chi))$. So we may interpret $H_T^*(\mathbb{P}(\chi))$ as a ring of piecewise polynomials $PP[\Sigma]$, where Σ has cones σ_i and Courant functions a_i, for $0 \leqslant i \leqslant n$.

According to relation (2·5), we may choose piecewise linear functions $f_0, \ldots, f_n$ in $H_T^2(\mathbb{P}(\chi))$ that are reduced, non-zero, and satisfy $f_0 \ldots f_n = 0$. On each cone σ_j, some f_i must therefore vanish; but it cannot vanish on σ_k for any other $k \neq j$, or else it would also vanish on every ray of Σ and be identically zero. Because f_i is reduced, it follows that $f_i = \pm a_j$. So we may assume that $f_i = \pm a_i$ for every $0 \leqslant i \leqslant n$, by permuting the cones as necessary.

Given any $i \neq j$, we may now read off the p-content q_{ij} of $a_i - a_j$ from the functions f_i, as follows. Since $f_i = \pm a_i$ and $f_j = \pm a_j$, we know that q_{ij} is the p-content of either $f_i - f_j$ or $f_i + f_j$; in fact it is the larger of the two (and the smaller is the p-content of $a_i + a_j$). This is because a_j restricts to 0 on σ_j, whence $(a_i - a_j)^{(j)} = (a_i + a_j)^{(j)}$. But $a_i - a_j$ is globally linear by Lemma 3·5, so its p-content is unaltered by restriction to σ_j, whereas that of $a_i + a_j$ may increase.

Appealing to Lemma 3·5 once more, we find that $\mathrm{lcm}(\chi_0, \ldots, \chi_n)/\mathrm{lcm}(\chi_i, \chi_j)$ has p-content q_{ij}. Moreover, there exist integers j and k for which $\mathrm{lcm}(\chi_j, \chi_k)$ is not divisible by p, since the weights are normalised. So for $i \neq j$ or k, the p-content of χ_i is precisely q_{jk}/q_{ik}, and the weights are completely determined up to order.

Remark 3. The analogue of Theorem 5·1 is false for ordinary cohomology, because the divisibility rule (1·4) does not take into account the relationship between the distribution of the p-contents of the weights for different primes p. For example, the graded rings $H^*(\mathbb{P}(1,2,3))$ and $H^*(\mathbb{P}(1,1,6))$ are isomorphic, since $c_1^2 = 6c_2$ in both cases. However, $\mathbb{P}(1,2,3)$ and $\mathbb{P}(1,1,6)$ cannot be homeomorphic, because the former has two singular points, and the latter only one; nevertheless, they are both homotopy equivalent to the 2-cell complex $S^2 \cup_{6\eta} e^4$, where η generates $\pi_3(S^2) \cong \mathbb{Z}$.

From the toric viewpoint, both quotient spaces $\mathbb{P}(1,2,3)/T$ and $\mathbb{P}(1,1,6)/T$ may be identified with the 2-simplex. On the other hand, it follows from Theorem 5·1 that the corresponding homotopy quotients cannot even be homotopy equivalent.

6. *Weighted projective bundles*

Suppose given complex line bundles L_i over a base space X for $0 \leqslant i \leqslant n$, and denote their direct sum by $D = L_0 \oplus \cdots \oplus L_n$. The torus $T' = (S^1)^{n+1}$ acts on D, and on the corresponding sphere bundle $S(D)$, in canonical fashion. The associated *weighted projective bundle* over X has fibre $\mathbb{P}(\chi)$, and total space the quotient

$$\mathbb{P}(D, \chi) = S(D)/S^1\langle \chi_0, \ldots, \chi_n \rangle. \tag{6·1}$$

The universal example is given by $E = ET' \times_{T'} \mathbb{C}^{n+1}$ over BT'. The reasoning of Lemma 2·1 shows that $H^*(\mathbb{P}(E, \chi))$ is a free $H^*(BT')$-module of rank $n+1$, and the naturality of the Serre spectral sequence implies the same result for arbitrary D.

If all weights are equal to 1, then $\mathbb{P}(D, \chi)$ is an ordinary projective bundle, and $H^*(\mathbb{P}(D, \chi))$ is generated by the Chern classes $c_1(L_i)$ and $-\xi$, subject only to the relation (1·5). In [**1**, chapter III], Al Amrani generalised (1·5) to those weighted projective bundles whose χ_i form a divisor chain. In all such cases, he proved that

$$\prod_{i=0}^{n}\left(\xi + \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_i}\, c_1(L_i)\right) = 0 \tag{6·2}$$

for a certain $\xi \in H^2(\mathbb{P}(D, \chi))$, which restricts to $c_1 \in H^2(\mathbb{P}(\chi))$ on fibres. By naturality, it suffices to verify (6·2) for the universal case; in other words, in $H^*_{T'}(\mathbb{P}(\chi))$, where T' acts on $\mathbb{P}(\chi)$ via the projection

$$T' \longrightarrow T = T'/S^1\langle \chi_0, \ldots, \chi_n\rangle. \tag{6·3}$$

We may profitably describe $H^*_{T'}(\mathbb{P}(\chi))$ in terms of piecewise polynomials, as follows.

Let $\pi\colon N' = \mathbb{Z}^{n+1} \to N$ be the epimorphism defined on canonical basis vectors by $\pi(e_i) = v_i$, for $0 \leqslant i \leqslant n$. Its kernel is the subgroup generated by $\chi_0 e_0 + \cdots + \chi_n e_n$, which we abbreviate to u. The pull-back $\Sigma' = \{\pi^{-1}(\sigma) : \sigma \in \Sigma\}$ of Σ is a *generalised fan* [**5**] because every cone contains the line through u, and π induces a monomorphism $\pi^*\colon PP[\Sigma] \to PP[\Sigma']$ of piecewise polynomial rings.

LEMMA 6·1. *As $H^*(BT')$-algebras, $H^*_{T'}(\mathbb{P}(\chi))$ is naturally isomorphic to $PP[\Sigma']$; furthermore, the projection $T' \to T$ induces the homomorphism $\pi^*\colon H^*_T(\mathbb{P}(\chi)) \to H^*_{T'}(\mathbb{P}(\chi))$.*

Proof. On cohomology rings we have the natural monomorphism

$$H^*_T(\mathbb{P}(\chi)) \longrightarrow H^*_{T'}(\mathbb{P}(\chi)) = H^*_T(\mathbb{P}(\chi)) \otimes H^*(BS^1), \tag{6·4}$$

because T' splits as $T \times S^1$, where S^1 is the kernel of (6·3) and acts trivially on $\mathbb{P}(\chi)$. So the freeness of $H^*_T(\mathbb{P}(\chi))$ over $H^*(BT)$ implies the freeness of $H^*_{T'}(\mathbb{P}(\chi))$ over $H^*(BT')$. Moreover, every isotropy subgroup of the T'-action is connected.

We can therefore imitate the reasoning of Proposition 2·2. For cones $\sigma' \in \Sigma'$ that have $k = \operatorname{codim} \sigma' \leqslant 1$ (and in fact for all cones σ'), we have that

$$H^{*+k}_{T'}(\bar{\mathcal{O}}_{\sigma'}, \partial\mathcal{O}_{\sigma'}) \cong \mathbb{Z}[\sigma'], \tag{6·5}$$

and the boundary map in cohomology corresponds to the restriction of polynomials up to sign. Hence the integral Chang–Skjelbred sequence is exact, which in turn identifies $H^*_{T'}(\mathbb{P}(\chi))$ with $PP[\Sigma']$.

Since for all $\sigma \in \Sigma$ and $\sigma' = \pi^{-1}(\sigma)$ the map

$$H^*_T(\bar{\mathcal{O}}_\sigma, \partial\mathcal{O}_\sigma) \longrightarrow H^*_{T'}(\bar{\mathcal{O}}_{\sigma'}, \partial\mathcal{O}_{\sigma'}) \tag{6·6}$$

corresponds to the pull-back of functions by π, the same applies to the restriction of equivariant cohomology from T to T'.

Because no cone of Σ' contains every e_i, we can define ξ as the piecewise linear function on N' which takes the values $\xi(u) = -\operatorname{lcm}(\chi_0, \ldots, \chi_n)$, and $\xi(e_i) = 0$ for all $0 \leqslant i \leqslant n$.

Equivalently,

$$\xi^{(i)} = -\frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_i}\, x_i \tag{6·7}$$

for all i, where (x_i) denotes the basis dual to (e_i) for N'.

THEOREM 6·2. *As an element of* $H^*_{T'}(\mathbb{P}(\chi))$*, the cohomology class* ξ *restricts to* c_1 *in* $H^2(\mathbb{P}(\chi))$ *and satisfies equation* (6·2).

Proof. As elements of $PP[\Sigma']$, we have that

$$\pi^*(a_i)^{(j)} = \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_i}\, x_i - \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_j}\, x_j \tag{6·8a}$$

$$= \frac{\operatorname{lcm}(\chi_0, \ldots, \chi_n)}{\chi_i}\, x_i + \xi^{(j)}, \tag{6·8b}$$

because the right-hand side vanishes on u and assumes the value $a_i(v_k)$ on e_k, for all k. Since ξ differs from $\pi^*(a_i)$ by a linear function, it restricts to the same element in $H^2(\mathbb{P}(\chi))$ as $\pi^*(a_i)$ and a_i, namely c_1. Identifying $c_1(L_i)$ with x_i, we conclude that equation (6·2) is nothing but the pull-back of relation (2·5).

No other relation is required to describe $H^*(\mathbb{P}(E, \chi))$ rationally.

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