Note on the Geometry of two Binary quadratics in the complex plane.

By W. SADDLER.

(*Read 14th June 1924. Received 30th June 1924*).

If
$$az^2 + 2bz + c = 0 \quad \ldots\ldots\ldots\ldots (1)$$
$$a'z^2 + 2b'z + c' = 0 \quad \ldots\ldots\ldots\ldots (2)$$

represent the points A, B and C, D respectively and if

$$ac' + a'c - 2bb' = 0$$

then A, B, C, D are four harmonic points on a circle. If O is the mid point of AB then $OA^2 = OC \,.\, OD$ and $\angle COB = \angle DOB$.*

To construct the Jacobian of these two quadratics, namely

$$(ab' - a'b)z^2 + (ac' - a'c)z + (bc' - b'c) = 0 \quad \ldots\ldots\ldots\ldots (3)$$

we proceed as follows.

The mid point of C, D being $-\dfrac{b'}{a'}$ has as its fourth harmonic with respect to (1) the point

$$z = \frac{bb' - ca'}{ba' - ab'}$$

so the mid point of A, B *i.e.* $-\dfrac{b}{a}$ has as its four harmonic with respect to (2) the point $z = \dfrac{ac' - bb'}{ba' - ab'}$.

The mid point of these two points just found will be $\frac{1}{2} \cdot \dfrac{ac' - a'c}{ba' - ab'}$ which is the mid point of the pair of points given by (3). We know that the pairs (1) and (2) are harmonic with respect to (3), and as we can thus determine the mid point of the pair (3) we can actually construct these points. Thus if M is this mid point : let RMS bisect the angle AMB. The perpendiculars to RS at M and

* §§ Grace and Young, *Algebra of Invariants*, pages 209, 210.

to AB at its mid point will meet at a point O which is the centre of the circle, with OA as radius, which will meet the line RS in the points represented by the Jacobian.

Taking the quadratic giving A, B as $z^2+pz+q=0$ and C, D as $z^2+p'z+q'=0$ we know that their Jacobian is harmonic to all points given by (4)...$\lambda(z^2+pz+q)+\mu(z^2+p'z+q')=0$, and in the case where λ, μ are real we can determine the mid point of the pair given by (4), and hence actually construct the points in this case.

A special result is the following.

Take any two points P, Q: the foci of the conic which touches the triangle ABP at the mid points of the sides being given by $\frac{df}{dz}=0$ where

$$f=(az^2+2bz+c)(z+p)=0 \quad \text{is}$$

$$3az^2+2z(2b+pa)+(c+2bp)=0$$

so for A, B, Q $\quad (az^2+2bz+c)(z+q)=0$ is

$$3az^2+2z(2b+qa)+(c+2bq)=0.$$

Let these points be $F_1 F_2 F_3 F_4$. The pair of points Harmonic to both $F_1 F_2$ and $F_3 F_4$ is given by

$$3a^2z^2+6abz+(4b^2-ac)=0$$

which is independent of P and Q: call these points M, N.

Treat the quadratic giving C, D in a similar manner and we get two points $M'N'$ given by $3a'^2z^2+6a'b'z+(4b'^2-a'c')=0$. The two points harmonic to M, N, M', N' are given by

$$4(ab'-a'b)\{aa'z^2+(ab'+a'b)z+a'b'\}$$
$$-aa'\{z^2(ab'-a'b)+(ac'-a'c)z+(bc'-b'c)\}=0.$$

The first part is the quadratic giving the mid points of the pair of points AB, CD, and the second part is the Jacobian (3).

§ 2. The second transvectant of the two cubics $a_z^3=0$ and $b_z^3=0$ was fixed geometrically by determining the *mid point* of the pair of points representing it, namely $(ab)^2 a_z b_z=0$, and fixing a pair of quadratics to which it is apolar.*

---

* *Proceedings Edin. Math. Society*, Vol. 42, Pt. I.

This mid point may be fixed as follows.

Consider the focal points of the first and second cubics, namely $a_0 z^2 + 2a_1 z + a_2 = 0 = a_z^2$ and $b_0 z^2 + 2b_1 z + b_2 = 0 = b_z^2$. If the roots of $a_z^2 = 0$ are y_1 and y_2: then determine the first polar of y_1 with respect to $b_z^3 = 0$, and then the first polar of y_2 with respect to the two points just found. This is the linear covariant of the quadratic

$$a_z^2 = 0 \text{ and } b_z^3 = 0$$

$$(ab)^2 b_z = 0 = \{z(a_2 b_0 - 2a_1 b_1 + b_2 a_0) + a_2 b_1 + a_0 b_3 - 2a_1 b_2 = 0\}.$$

Treating similarly $b_z^2 = 0$ and $a_z^3 = 0$ we get

$$(ab)^2 a_z = 0 = \{z(a_2 b_0 - 2a_1 b_1 + b_2 a_0) + b_2 a_1 + a_3 b_0 - 2a_2 b_1 = 0\}.$$

The mid point of these two points is the mid point of the second transvectant $(ab)^2 a_z b_z = 0$.

Incidentally if *the two points so found coincide* we get an interpretation for the vanishing of the $(ab)^3$ or the Apolarity condition for the two cubics, $a_z^3 = 0$ $b_z^3 = 0$.

i.e.
$$-\frac{b_2 a_1 + a_3 b_0 - 2a_2 b_1}{a_2 b_0 - 2a_1 b_1 + b_2 a_0} = -\frac{a_2 b_1 + a_0 b_3 - 2a_1 b_2}{a_2 b_0 - 2a_1 b_1 + b_2 a_0}$$

or
$$a_0 b_3 - 3a_1 b_2 + 3a_2 b_1 - a_3 b_0 = 0.$$