

Again, from similar triangles,

$$\frac{AL^2}{DK^2} = \frac{FL^2}{DF^2} = \frac{CE^2}{CD^2} = \frac{FL^2 - CE^2}{DF^2 - CD^2} = \frac{FE^2 - LC^2}{DF^2 - CD^2}$$

$$= \frac{AC^2 - LC^2}{DF^2 - CD^2}$$

$$\therefore \left. \begin{aligned} DK^2 &= DF^2 - CD^2 \\ \text{i.e., } DK &= \sqrt{s(s-a)} \end{aligned} \right\} \dots\dots\dots (ii)$$

From (i) and (ii) it is clear that

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

2. $AD^2 + CD^2 = \frac{b^2 + c^2}{2} = DE^2 + DF^2$

$$\therefore DK^2 = DF^2 - CD^2 = AD^2 - DE^2$$

3. The above analysis enables us to devise simple geometrical constructions for triangles, given the sum or difference of two sides and any two of the three lengths: the base, the altitude perpendicular to the base, and the median bisecting the base.

Note II.—The Ambiguous Case.

The following two converses of the Ambiguous Case are noteworthy:—

Converse (1) If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles opposite to one pair of equal sides supplementary, then the angles opposite to the other pair are equal.

The case of congruence of triangles which agree as to two sides and a right angle not included, is a particular case of the above.

The corresponding theorem in similarity may be enunciated thus:

Theorem: If two triangles have two sides of the one proportional to two sides of the other and the angles opposite to one pair of corresponding sides supplementary, then the angles opposite to the other pair are equal.

(a) An immediate inference from the above theorem is Euclid VI. 3. For, if AD divide the base BC of a triangle ABC internally in the ratio $AB : AC$, from the triangles ABD , ACD , we derive from our theorem $\widehat{BAD} = \widehat{CAD}$.

(b) Another theorem that can be easily deduced is the converse of a well-known theorem, not I believe previously noted by any geometrician.

If a circle be circumscribed about a triangle and the rectangle contained by two sides of the triangle be equal to that contained by two straight lines issuing from the vertex, one terminated by the base or that produced and the other by the circumference, then these two straight lines either make equal angles with the sides or include between them an angle equal to the difference of the base angles. (The proof being easy is left to the reader.)

Converse (2). If two triangles have a pair of angles equal and another pair supplementary each to each, and also the sides opposite to one pair of these angles equal, then the sides opposite to the other pair of angles are also equal.

The proof easily follows by suitable partial superposition of the triangles.

Two interesting applications of the theorem may be given here :

(i) ABC is a triangle ($AB > AC$) and the bisector of the angle A meets BC in D . To prove that $BD > CD$.

From BA cut off $BF = AC$ and draw FE parallel to AD to meet BC in E . Then from the triangles BFE , CAD , by our theorem, $BE = DC$.

$$\therefore BD > CD.$$

Similarly we can prove that if $BD > DC$, $AB > AC$.

(ii) ABC , $A'B'C'$ are two triangles such that $AB = A'B'$, $BC = B'C'$. To prove that the middle points of AA' , BB' , CC' are in a straight line.

Let M , N , P be the middle points of AA' , BB' , CC' respectively. Through A , A' draw AL , $A'L'$ parallel to the bisector of the angle between AC and $A'C'$ to meet BB' at L , L' respectively.

Then, from the triangles ABL , $A'B'L'$, by our theorem, $BL = B'L'$.

$\therefore N$ is the middle point of LL' and MN is parallel to AL or $A'L'$, i.e. to the bisector of the angle between AC and $A'C'$. Similarly we may show that MP is also parallel to the same bisector.

Hence M, N, P are collinear.

The theorem in similarity corresponding to the converse theorem (2) is the following:

If two triangles have an angle of the one equal to an angle of the other and another pair of angles supplementary, then the sides opposite to the equal angles are proportional to those opposite to the supplementary angles.

Thus, in Euclid VI. 3 again, if AD bisect the angle BAC internally or externally, it readily follows from the triangles ADB, ADC (where one pair of angles are equal and another pair supplementary) that

$$AB : AC = BD : DC.$$

Note III.—Simson-Line.

The following is a new proof of the well-known theorem that “the pedal line of a point with respect to a triangle inscribed in a circle passing through the point, bisects the join of the point and the orthocentre of the triangle.”

