

The condition that the family of surfaces $\phi(x, y, z, \lambda) = 0$ may be one of an orthogonal triad.

By F. E. EDWARDES.

(Received 7th October 1911. Read 12th January 1912).

Cayley has shown that if the family of surfaces $\phi(x, y, z) = \lambda$ is one of an orthogonal triad, ϕ satisfies a differential equation of the third order. If, however, the parameter λ is involved implicitly in the equation of the family, the condition requires modification: for example, although

$$\phi \equiv \frac{x^2}{a + \lambda} + \frac{y^2}{b + \lambda} + \frac{z^2}{c + \lambda} - 1 = 0$$

is one of an orthogonal triad, ϕ does not satisfy Cayley's equation.

In a previous paper* I have discussed a method of obtaining Cayley's equation. The first modification that is necessary occurs on p. 49. The condition that C may lie on the consecutive surface is now

$$\phi(x + L\rho, y + M\rho, z + N\rho, \lambda + d\lambda) = 0$$

or $R^2\rho + \phi'd\lambda = 0$,

ϕ' denoting $\frac{\partial\phi}{\partial\lambda}$. Thus $R^2\rho/\phi'$ is constant as we proceed along the surface λ , and so

$$\frac{d\rho}{\rho} + \frac{d(R^2)}{R^2} - \frac{d\phi'}{\phi'} = 0,$$

where

$$\begin{aligned} d\phi' &= \left(dx \frac{\partial}{\partial x} + dy \frac{\partial}{\partial y} + dz \frac{\partial}{\partial z} \right) \frac{\partial\phi}{\partial\lambda} = dx \frac{\partial L}{\partial\lambda} + dy \frac{\partial M}{\partial\lambda} + dz \frac{\partial N}{\partial\lambda} \\ &= dxL' + dyM' + dzN'. \end{aligned}$$

* *Proc. Edin. Math. Soc.*, vol. xxix.

Also the value of a at the point C is

$$a + \rho \delta a + \frac{\partial a}{\partial \lambda} d\lambda$$

or

$$a + \rho \left(\delta a - \frac{R^2 a'}{\phi'} \right).$$

In the conjugate relation on the consecutive surface (p. 50) we must therefore replace δa by $\delta a - \frac{R^2 a'}{\phi'}$ and so on, and add new terms arising from the parts of $d_1 \rho$ and $d_2 \rho$ which involve ϕ' . The additional terms are

$$\frac{d_1 \phi'}{\phi'} (d_2 x \delta L + d_2 y \delta M + d_2 z \delta N) + \frac{d_2 \phi'}{\phi'} (d_1 x \delta L + d_1 y \delta M + d_1 z \delta N)$$

$$\text{i.e. } \frac{1}{\phi'} \{ (d_1 x L' + d_1 y M' + d_1 z N') (d_2 x \delta L + d_2 y \delta M + d_2 z \delta N) \\ + (d_2 x L' + d_2 y M' + d_2 z N') (d_1 x \delta L + d_1 y \delta M + d_1 z \delta N) \}$$

$$\text{i.e. } d_1 x d_2 x \frac{2L' \delta L}{\phi'} + \dots + \dots + (d_1 y d_2 z + d_2 y d_1 z) \frac{M' \delta N + N' \delta M}{\phi'} + \dots + \dots$$

The other equations involving $d_1 x d_2 x$, etc., are unchanged, so that the first row of the modified determinant is

$$\delta a - 2A - \frac{R^2 a'}{\phi'} + \frac{2L' \delta L}{\phi'}, \dots, \dots, 2 \left(\delta f' - 2F - \frac{R^2 f'}{\phi'} + \frac{M' \delta N + N' \delta M}{\phi'} \right), \dots, \dots,$$

and the other rows are the same as before.

Hence the condition that $\phi(x, y, z, \lambda) = 0$ may be a family belonging to an orthogonal triad is

$$\mathfrak{A} \left(\delta a - 2A - \frac{R^2 a'}{\phi'} + \frac{2L' \delta L}{\phi'} \right) + \dots + \dots \\ + 2 \mathfrak{F} \left(\delta f' - 2F - \frac{R^2 f'}{\phi'} + \frac{M' \delta N + N' \delta M}{\phi'} \right) + \dots + \dots = 0.$$