

A SPECTRAL MAPPING THEOREM FOR SOME REPRESENTATIONS OF COMPACT ABELIAN GROUPS

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Dedicated to Professor Chinami Watari on his sixtieth birthday

We show that if G is a compact abelian group and U is a weakly continuous representation of G by means of isometries on a Banach space X , then $\sigma(\pi(\mu)) = \widehat{\mu}(sp(U))$ holds for each measure μ in $\text{reg}(M(G))$, where $\pi(\mu)$ denotes the generalized convolution operator in $B(X)$ defined by $\pi(\mu)x = \int_G U(t)x d\mu(t)$ ($x \in X$), σ the usual spectrum in $B(X)$, $sp(U)$ the Arveson spectrum of U , $\widehat{\mu}$ the Fourier–Stieltjes transform of μ and $\text{reg}(M(G))$ the largest closed regular subalgebra of the convolution measure algebra $M(G)$ of G . $\text{reg}(M(G))$ contains all the absolutely continuous measures and discrete measures.

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1. Introduction and main result

Let G be a locally compact abelian group and U a weakly continuous representation of G by means of isometries on a Banach space X , i.e., a map $U: G \rightarrow B(X)$ satisfying

- (i) $U(s+t) = U(s)U(t)$ for all $s, t \in G$, $U(0) = I$,
- (ii) $\|U(s)x\| = \|x\|$ for $s \in G$, $x \in X$,
- (iii) $G \rightarrow X; s \rightarrow U(s)x$ is weakly continuous for each $x \in X$.

Then this representation induces a continuous algebra homomorphism π of the convolution algebra $M(G)$ into $B(X)$ and such a homomorphism is written by $\pi(\mu) = \int_G U(t) d\mu(t)$ (cf. [5]). Let $sp(U)$ be the Arveson spectrum of U defined by

$$sp(U) = \bigcap \{Z(f) : f \in \text{Ker}(\pi|L^1(G))\}.$$

Here $Z(f)$ denotes the set of zeros of the Fourier transform \hat{f} of f . In this setting, Connes [4] proved that for every Dirac measure μ the spectral mapping theorem (SMT): $\sigma(\pi(\mu)) = \widehat{\mu}(sp(U))$ holds, where σ denotes the usual spectrum in $B(X)$ and $\widehat{\mu}$ denotes the Fourier–Stieltjes transform of μ . Furthermore D’Antoni, Longo and Zsido [2] proved the SMT for the class of measures whose continuous part belongs to $L^1(G)$, the group algebra of G . Also, Eschmeier [5] proved the SMT in the case that U is the translation group representation and $X = L^1(G)$ or $M(G)$ and the convolution operator induced by μ has the weak 2-SDP (see [5, Theorem 2]). Here $M(G)$ denotes the Banach

algebra of all bounded regular complex Borel measures on G . Also, since $M(G)$ is a semisimple commutative Banach algebra with identity, it follows from Albrecht's theorem [1] that there exists a largest closed regular subalgebra of $M(G)$, which we denote by $\text{reg}(M(G))$.

With this notation, our main theorem can be stated as follows:

Theorem. *If G is a compact abelian group and $\mu \in \text{reg}(M(G))$, we have $\sigma(\pi(\mu)) = \hat{\mu}(sp(U))$.*

Remark. The group algebra $L^1(G)$ and the discrete measures $M_d(G)$ are regular Banach subalgebras of $M(G)$. Then $L^1(G) + M_d(G) \subset \text{reg}(M(G))$. In general, $L^1(G) + M_d(G) \neq \text{reg}(M(G))$. In fact, let us denote by $\text{top}(G)$ the class of all locally compact group topologies on G which are equal to or stronger than the original topology on G and denote by $L^*(G)$ the closed subalgebra of $M(G)$ generated by $\{L^1(G, \tau) : \tau \in \text{top}(G)\}$ as in [6]. Then we have that $L^1(G) + M_d(G) \subset L^*(G) \subset \text{reg}(M(G))$ and $L^1(G) + M_d(G) \neq L^*(G)$ in general (cf. [6], [12]). Thus our result contains the Connes–D'Antoni–Longo–Zsido spectral mapping theorem for the compact case.

2. Lemmas

We first present the following result obtained in [7], which plays an essential role in the proof of the main theorem, and we include its proof for completeness.

Lemma 1. *Let X be a commutative Banach algebra with identity and B a Banach subalgebra of X . If B is regular, then for any $b \in B$ the Gelfand transform of b as an element of X is continuous on the carrier space Φ_X of X in the hull–kernel topology.*

Proof. We can assume without loss of generality that B contains the identity of X . Then it is sufficient to show that the restriction map $\theta: \Phi_X \rightarrow \Phi_B; \phi \rightarrow \phi|_B$ is continuous in the hull–kernel topology. To do this let F be a closed subset of Φ_B in the hull–kernel topology. Then $\{\phi \in \Phi_X : \phi|_{\ker F} = 0\} = \theta^{-1}(F)$. Also, since $\ker F \subset \ker \theta^{-1}(F)$, it follows that $\text{hul}(\ker \theta^{-1}(F)) \subset \{\phi \in \Phi_X : \phi|_{\ker F} = 0\}$. Therefore $\theta^{-1}(F)$ is closed in the hull–kernel topology. In other words, θ is continuous in this topology. \square

We will next state the definition of BSE-algebras introduced by the first author and Hatori [10]. Let A be a commutative Banach algebra without order and $M(A)$ the multiplier algebra of A . It is well-known that $T \in M(A)$ can be represented as a bounded continuous complex-valued function \hat{T} on Φ_A such that $\widehat{Ta}(\phi) = \hat{T}(\phi)\hat{a}(\phi)$ for all $a \in A$ and $\phi \in \Phi_A$ (cf. [8]). Set $\hat{M}(A) = \{\hat{T} : T \in M(A)\}$. We also denote by A^* the dual space of A and $C_{\text{BSE}}(\Phi_A)$ the set of all continuous complex-valued functions σ on Φ_A which satisfy the following condition: there exists a positive real number β such that for every finite sequence of complex numbers c_1, \dots, c_n and elements ϕ_1, \dots, ϕ_n of Φ_A , the inequality

$$\left| \sum_{i=1}^n c_i \sigma(\phi_i) \right| \leq \beta \left\| \sum_{i=1}^n c_i \phi_i \right\|_{A^*}$$

holds.

Definition. A commutative Banach algebra A without order is said to be BSE if it satisfies the condition $\widehat{M}(A) = C_{\text{BSE}}(\Phi_A)$.

By the Bochner–Schoenberg–Eberlein theorem, the group algebra of a locally compact abelian group is BSE (cf. [10]). The following results can be observed in [10].

Lemma 2 ([10, Theorem 4, (ii)]). *Let A be a commutative Banach algebra without order, A^{**} its second dual and $C^b(\Phi_A)$ the set of all bounded continuous complex-valued functions on Φ_A . Then $C_{\text{BSE}}(\Phi_A) = C^b(\Phi_A) \cap (A^{**}|_{\Phi_A})$.*

When a closed ideal I of a commutative Banach algebra A is essential as a Banach A -module, that is, I equals the closed linear span of $\{ax : a \in A, x \in I\}$, we call I an essential ideal.

Lemma 3 ([10, Theorem 8, (i)]). *Let A be a BSE-algebra with discrete carrier space and I an essential closed ideal of A . Then $\widehat{M}(A/I) = C_{\text{BSE}}(\Phi_{A/I})$, i.e., A/I is BSE, where A/I denotes the quotient algebra of A defined by I .*

The following lemma also plays an essential role in the proof of the main theorem.

Lemma 4. *Let A be a BSE-algebra with discrete carrier space and I a closed ideal of A such that $I^\sim = \ker(\text{hul}(I))$ is essential. Then every multiplier on A/I^\sim can be lifted as a multiplier on A , that is, if $v \in M(A/I^\sim)$ and η is the canonical map of A onto A/I^\sim , then there exists $\mu \in M(A)$ such that $\eta(\mu a) = v\eta(a)$ for all $a \in A$.*

Proof. Let $v \in M(A/I^\sim)$ and let η be the canonical map of A onto A/I^\sim . Note that the algebra A/I^\sim is semisimple. Then it is sufficient to show that there exists $\mu \in M(A)$ such that $(\eta(\mu a))^\wedge = (v\eta(a))^\wedge$ for all $a \in A$. Here \wedge denotes the Gelfand transform on A/I^\sim . We have

$$\begin{aligned} \widehat{v} \in \widehat{M}(A/I^\sim) &= C_{\text{BSE}}(\Phi_{A/I^\sim}) && \text{(by Lemma 3)} \\ &= (A/I^\sim)^{**}|_{\Phi_{A/I^\sim}} && \text{(by Lemma 2),} \end{aligned}$$

so that there exists $H \in (A/I^\sim)^{**}$ with $\widehat{v} = H|_{\Phi_{A/I^\sim}}$. Then we can find an element $F \in A^{**}$ such that $\eta^{**}(F) = H$, since $\eta^{**}: A^{**} \rightarrow (A/I^\sim)^{**}$ is a surjection. We further have by the BSE property of A and Lemma 2 that

$$\widehat{M}(A) = C_{\text{BSE}}(\Phi_A) = A^{**}|_{\Phi_A}.$$

Therefore we can find an element $\mu \in M(A)$ such that $\widehat{\mu} = F|_{\Phi_A}$. Let $\phi \in \Phi_A$ be such that $\phi|_{I^\sim} = 0$ and ϕ' the canonical image of ϕ in Φ_{A/I^\sim} . Then we have

$$\hat{v}(\phi') = H(\phi') = \langle \phi', \eta^{**}(F) \rangle = \langle \eta^*(\phi'), F \rangle = \langle \phi, F \rangle$$

and hence for any $a \in A$,

$$\begin{aligned} (v\eta(a))^\wedge(\phi') &= \hat{v}(\phi')(\eta(a))^\wedge(\phi') = \langle \phi, F \rangle \hat{a}(\phi) \\ &= \hat{\mu}(\phi) \hat{a}(\phi) = (\mu a)^\wedge(\phi) = \phi'(\eta(\mu a)) \\ &= (\eta(\mu a))^\wedge(\phi'). \end{aligned}$$

Consequently $(\eta(\mu a))^\wedge = (v\eta(a))^\wedge$ for all $a \in A$. \square

Lemma 5. *If $\mu \in M(G)$, then $\sigma(\pi(\mu)) \subset \{\mu^\vee(\phi) : \phi \in \Phi_{M(G)}, \text{Ker } \pi \subset \text{Ker } \phi\}$. Here μ^\vee denotes the Gelfand transform of $\mu \in M(G)$.*

Proof. Let $\mu \in M(G)$. Then we have that

$$\begin{aligned} \sigma_{M(G)/\text{Ker } \pi}(\mu + \text{Ker } \pi) &= (\mu + \text{Ker } \pi)^\vee(\Phi_{M(G)/\text{Ker } \pi}) \\ &= \{\mu^\vee(\phi) : \text{Ker } \pi \subset \text{Ker } \phi\}. \end{aligned}$$

Also, since $M(G)/\text{Ker } \pi \cong \pi(M(G)) \subset B(X)$, it follows that

$$\begin{aligned} \sigma_{M(G)/\text{Ker } \pi}(\mu + \text{Ker } \pi) &= \sigma_{\pi(M(G))}(\pi(\mu)) \\ &\supset \sigma_{B(X)}(\pi(\mu)). \end{aligned}$$

Therefore the desired inclusion follows. \square

The following result was proved by D'Antoni, Longo and Zsido [2].

Lemma 6 ([2, Lemma 1]). $\sigma(\pi(\mu)) \supset \overline{\hat{\mu}(sp(U))}$ for all $\mu \in M(G)$.

In the next section we will show our main theorem using these lemmas.

3. Proof of theorem

Since $\overline{\hat{\mu}(sp(U))} \subset \sigma(\pi(\mu))$ by Lemma 6, we have only to show the reverse inclusion. To do this, let $\alpha \in \sigma(\pi(\mu))$. Then by Lemma 5, there exists $\phi_0 \in \Phi_{M(G)} : \alpha = \mu^\vee(\phi_0)$ and $\text{Ker } \pi \subset \text{Ker } \phi_0$.

Let us consider the natural homomorphism T_π of $M(G)/\text{Ker } \pi$ into $M(L^1(G)/I_\pi)$ defined by

$$T_\pi(v + \text{Ker } \pi)(f + I_\pi) = v * f + I_\pi \quad (v \in M(G), f \in L^1(G)),$$

where $I_\pi = \text{Ker}(\pi|_{L^1(G)})$.

Since G is compact, it follows from [9, Corollary 8.3.2] that $I_\pi \sim I_\pi$. Note also that $L^1(G)$ is a BSE-algebra with discrete carrier space and it has an approximate identity; hence I_π is an essential ideal of $L^1(G)$. Then Lemma 4 implies that T_π is surjective, since $M(G) \cong M(L^1(G))$. Furthermore, T_π is injective. In fact, let $v \in M(G)$ be such that $\pi(v * f) = 0$ for all $f \in L^1(G)$. Given $\varepsilon > 0$, $x \in X$ and $\xi \in X^*$, the dual space of X , choose a neighbourhood V of zero such that

$$|\langle U(t)x, (\pi(v))^*\xi \rangle - \langle x, (\pi(v))^*\xi \rangle| < \varepsilon \quad (t \in V).$$

Furthermore, choose a non-negative real-valued function $u_V \in L^1(G)$ vanishing off V and satisfying $\int_G u_V(t) dt = 1$. Then we have

$$\begin{aligned} |\langle \pi(v)x, \xi \rangle| &\leq |\langle \pi(u_V)x, (\pi(v))^*\xi \rangle - \langle x, (\pi(v))^*\xi \rangle| + |\langle \pi(u_V)x, (\pi(v))^*\xi \rangle| \\ &\leq \int_V |\langle U(t)x, (\pi(v))^*\xi \rangle - \langle x, (\pi(v))^*\xi \rangle| u_V(t) dt \\ &< \varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that $\langle \pi(v)x, \xi \rangle = 0$ for all $x \in X$ and $\xi \in X^*$; hence $\pi(v) = 0$. In other words, T_π is injective.

Here we take the following convention: for each $\phi \in \Phi_{M(G)}$ such that $\text{Ker} \pi \subset \text{Ker} \phi$, ϕ' denotes the element of $\Phi_{M(G)/\text{Ker} \pi}$ defined by $\phi'(v + \text{Ker} \pi) = \phi(v)$ ($v \in M(G)$).

Since T_π is an isomorphism of $M(G)/\text{Ker} \pi$ onto $M(L^1(G)/I_\pi)$, there exists an element ψ_0 of $\Phi_{M(L^1(G)/I_\pi)}$ such that $\phi'_0 = (T_\pi)^*\psi_0$. So we can find a net $\{\psi_\lambda\}$ in $\Phi_{M(L^1(G)/I_\pi)}$ such that $\psi_\lambda|_{L^1(G)/I_\pi} \neq 0$ for all λ and $hk\text{-lim} \psi_\lambda = \psi_0$, where “ $hk\text{-lim}$ ” denotes the hull–kernel limit. Furthermore, we can find a net $\{\phi_\lambda\}$ in $\Phi_{M(G)}$ such that $\text{Ker} \pi \subset \text{Ker} \phi_\lambda$ and $(T_\pi)^*\psi_\lambda = \phi'_\lambda$ for all λ . Set $\xi_\lambda = \phi_\lambda|_{L^1(G)}$ for each λ . Then each $\xi_\lambda \neq 0$. In fact, choose a function $f_0 \in L^1(G)$ such that $\psi_\lambda(f_0 + I_\pi) \neq 0$. Then for each λ , we have

$$\phi_\lambda(f_0) = \phi'_\lambda(f_0 + I_\pi) = \langle T_\pi(f_0 + I_\pi), \psi_\lambda \rangle = \psi_\lambda(f_0 + I_\pi) \neq 0,$$

so that $\xi_\lambda \neq 0$. Thus each ξ_λ belongs to $\Phi_{L^1(G)}$ ($\cong \widehat{G}$, the dual group of G) and hence must belong to $sp(U)$, since $I_\pi \subset \text{Ker} \xi_\lambda$ and $sp(U)$ can be regarded as the hull of I_π in $\Phi_{L^1(G)}$.

Of course $(T_\pi)^*|_{\Phi_{M(L^1(G)/I_\pi)}}$ is continuous on $\Phi_{M(L^1(G)/I_\pi)}$ in the hull–kernel topology and hence

$$hk\text{-lim} \phi'_\lambda = hk\text{-lim} (T_\pi)^*\psi_\lambda = (T_\pi)^*\psi_0 = \phi'_0.$$

Therefore we have from [11, Theorem 2.6.6] that $hk\text{-lim} \phi_\lambda = \phi_0$. Since $\mu \in \text{reg}(M(G))$, it follows from Lemma 1 that μ^\vee is continuous on $\Phi_{M(G)}$ in the hull–kernel topology, so that

$$\lim \hat{\mu}(\xi_\lambda) = \lim \mu^\vee(\phi_\lambda) = \mu^\vee(\phi_0) = \alpha.$$

Consequently we have that $\alpha \in \overline{\hat{\mu}(sp(U))}$ and the reverse inclusion is shown. \square

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