

# THE RADON-NIKODYM THEOREM FOR MULTIMEASURES

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## 1. Introduction

Let  $(S, \mathcal{M})$  be a *measurable space* (that is, a set  $S$  in which is defined a  $\sigma$ -algebra  $\mathcal{M}$  of subsets) and  $X$  a locally convex space. A map  $M$  from  $\mathcal{M}$  to the family of all non-empty subsets of  $X$  is called a *multimeasure* iff for every sequence of disjoint sets  $A_n \in \mathcal{M}$  ( $n = 1, 2, \dots$ ) with  $\bigcup_{n=1}^{\infty} A_n = A$ , the series  $\sum_{n=1}^{\infty} M(A_n)$  converges (in the sense of (6), p. 3) to  $M(A)$ .

The concept of multimeasure with values in  $\mathbb{R}^n$  was first introduced by Vind (15, p. 174) in order to solve some problems in economics. In (14, Théorème 23, p. 292), Valadier has proved the Radon-Nikodym theorem for multimeasures taking values in  $\mathbb{R}^n$  (or  $\mathbb{R}^{\infty}$ ) using the notion of scalar integrability of set-valued functions. (For further results in this aspect, see (3) and (11).)

In Section 2 of this paper, we shall define integrability for a special class of set-valued functions, which we shall call *perfectly measurable multifunctions*. Then we prove a theorem (Theorem 1) that serves as an example of a multimeasure. In Section 3 we prove the main result, the Radon-Nikodym theorem for multimeasures taking values in a locally convex space; this, however, is not a generalisation of Théorème 23 of (14), nor a consequence of that.

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## 2. Integrable multifunctions

Henceforth,  $(S, \mathcal{M})$  is a measurable space,  $\mu$  is a finite positive measure on  $\mathcal{M}$  and  $X$  is a Hausdorff locally convex space with (topological) dual  $X'$ , except where otherwise specified.

Let  $F$  be a map that assigns to each  $t \in S$ , a non-empty set  $F(t) \subseteq X$ . Then  $F$  is called a *multifunction* (or a set-valued function) from  $S$  to  $X$ . A point-valued function  $f$  from  $S$  to  $X$  is called a *selector* for  $F$  iff  $f(t) \in F(t)$  for every  $t \in S$ . For any subset  $B$  of  $X$ , we put

$$F^{-1}(B) = \{t \in S : F(t) \cap B \neq \emptyset\}.$$

The multifunction  $F$  is called *measurable* iff  $F^{-1}(B) \in \mathcal{M}$  for each closed subset  $B$  of  $X$ . We say that  $F$  is *perfectly measurable* iff it is measurable and, for every closed subset  $B$  of  $X$ , the multifunction  $F_B$  (called the *refinement* of  $F$  by  $B$ ), defined on  $F^{-1}(B)$  by  $F_B(t) = F(t) \cap B$ , has a measurable selector.

The following Lemma 1 that assures a measurable selector for  $F$  is due to Leese

(7) (for other results on the existence of measurable selectors for a multifunction, we refer to (1), (4), (10) and (12)). Because this work does not yet seem to have been published, a brief proof is included.

**Lemma 1.** *Suppose that  $X'$  contains a sequence  $(x'_n)$ ,  $n = 1, 2, \dots$ , which separates the points of  $X$ . Then every compact-valued measurable multifunction  $F$  from  $S$  to  $X$  has a measurable selector.*

**Proof.** For each  $t \in S$ , let  $F_0(t) = F(t)$  and define  $F_n(t)$  ( $n = 1, 2, \dots$ ) inductively as follows

$$F_n(t) = \{x \in F_{n-1}(t) : \langle x, x'_n \rangle \text{ maximal}\}.$$

Then it can be shown that each  $F_n$  is a compact-valued measurable multifunction from  $S$  to  $X$ . Moreover, it is clear that  $\bigcap_{n=1}^{\infty} F_n(t)$  consists of a single point,  $f(t)$  say, and that for every closed set  $B$  in  $X$ ,

$$f^{-1}(B) = \bigcap_{n=1}^{\infty} F_n^{-1}(B).$$

Therefore  $f$  is a measurable selector for  $F$ .

**Lemma 2.** *Suppose that  $X'$  contains a sequence which separates the points of  $X$ . Then every compact-valued measurable multifunction  $F$  from  $S$  to  $X$  is perfectly measurable.*

**Proof.** Let  $B$  be a closed subset of  $X$ . Let  $F_B$  be the refinement of  $F$  by  $B$ , which is defined on  $F^{-1}(B)$  by  $F_B(t) = F(t) \cap B$ . Then for every closed set  $C$  in  $X$ ,

$$F_B^{-1}(C) = F^{-1}(B \cap C),$$

which is measurable. Thus  $F_B$  is a compact-valued measurable multifunction from  $F^{-1}(B)$  to  $X$ . Hence, by Lemma 1,  $F_B$  has a measurable selector. Therefore  $F$  is perfectly measurable.

Before going on, let us recall that a (point-valued) function  $f$  from  $(S, \mathcal{M}, \mu)$  to  $X$  is called *scalarly integrable* iff for every  $x' \in X'$ , the function  $\langle x', f \rangle = x' \circ f$  is integrable. Then, for any  $A \in \mathcal{M}$ , we denote by  $\int_A f d\mu$  the linear form on  $X'$  defined by

$$\left\langle x', \int_A f d\mu \right\rangle = \int_A \langle x', f \rangle d\mu.$$

A measurable function  $f$  from  $(S, \mathcal{M}, \mu)$  to  $X$  is said to be *integrable* iff  $f$  is scalarly integrable and  $\int_A f d\mu \in X$  for every  $A \in \mathcal{M}$  (see for example (9)).

Now let  $F$  be a multifunction from  $(S, \mathcal{M}, \mu)$  to  $X$  and let  $\mathcal{S}(F)$  denote the set of all measurable selectors of  $F$ . The multifunction  $F$  is called *integrable* iff  $F$  is perfectly measurable and every  $f \in \mathcal{S}(F)$  is integrable. We denote, for any  $A \in \mathcal{M}$ ,

$$\int_A F d\mu = \left\{ \int_A f d\mu : f \in \mathcal{S}(F) \right\},$$

which is a subset of  $X$ . Note that we require  $F$  to be perfectly measurable so that

every refinement of  $F$  (by any closed subset  $B$  of  $X$ ) contributes to the integral (of course provided that  $A \cap F^{-1}(B)$  has a non-zero measure). Otherwise, it may happen that  $F(t) = G(t) \cup \{x\}$  for each  $t \in S$ , where  $G(t)$  is contained in a fixed closed subset  $B$  of  $X$ ,  $G$  has no measurable selector, and  $x \in X \setminus B$ . In such a case, the integral of  $F$  does not reflect the full range of values taken by  $F$  at all (For the basic properties of integrable multifunctions, see (13)).

**Theorem 1.** *Let  $F$  be an integrable multifunction from  $S$  to  $X$ . Then the set-valued map  $M$  from  $\mathcal{M}$  to  $X$ , defined by*

$$M(A) = \int_A F d\mu \quad (A \in \mathcal{M}),$$

*is a multimeasure.*

**Proof.** Let  $(A_n)$ ,  $n = 1, 2, \dots$ , be a sequence of disjoint sets in  $\mathcal{M}$  and let  $A = \bigcup_{n=1}^{\infty} A_n$ . We prove that

$$M(A) = \sum_{n=1}^{\infty} M(A_n).$$

For each  $n$ , let  $x_n \in M(A_n)$ . Then there exist  $f_n \in \mathcal{S}(F)$  such that  $x_n = \int_{A_n} f_n d\mu$ ,  $n = 1, 2, \dots$ . Let us define a function  $f$  by

$$f = \begin{cases} f_n & \text{on } A_n, \quad n = 1, 2, \dots \\ f_1 & \text{on } S \setminus A. \end{cases}$$

Certainly  $f \in \mathcal{S}(F)$ , hence  $f$  is integrable and  $x_n = \int_{A_n} f d\mu$ . Now, for every  $x' \in X'$  and every positive integer  $N$ ,

$$\left\langle x', \sum_{n=1}^N x_n \right\rangle = \sum_{n=1}^N \int_{A_n} \langle x', f \rangle d\mu,$$

which converges, as  $N \rightarrow \infty$ , to

$$\int_A \langle x', f \rangle d\mu = \left\langle x', \int_A f d\mu \right\rangle.$$

This means that the series  $\sum_{n=1}^{\infty} x_n$  converges weakly to  $x = \int_A f d\mu$ , and a similar property holds for every subseries of  $\sum_{n=1}^{\infty} x_n$ . Hence, by the Orlicz-Pettis Theorem (see for example (6), p. 4), the series  $\sum_{n=1}^{\infty} x_n$  converges (unconditionally) to  $x$ , which belongs to  $M(A)$ . Thus we have proved that the series  $\sum_{n=1}^{\infty} M(A_n)$  is (unconditionally) convergent and is contained in  $M(A)$ .

To prove the reverse inclusion let  $x \in M(A)$ ; then  $x = \int_A f d\mu$  for some  $f \in \mathcal{S}(F)$ . Then, as before, the series  $\sum_{n=1}^{\infty} \int_{A_n} f d\mu$  converges to  $x$ . This shows that  $x \in \sum_{n=1}^{\infty} M(A_n)$ , and completes the proof.

### 3. The Radon-Nikodym theorem

Let  $K$  be a convex closed subset of  $X$  and let  $x' \in X'$ . We write  $\varphi(x', K) = \sup\{\langle x', x \rangle : x \in K\}$ . Following Meyer (8, p. 32), we denote by  $\mathcal{L}^{\infty}(S, \mathcal{M})$  (resp.  $\mathcal{L}^1(S, \mathcal{M}, \mu)$ ) the vector space of all measurable bounded (resp. integrable) real-valued

functions on  $S$  and by  $L^\sigma(S, \mathcal{M}, \mu)$  (resp.  $L^1(S, \mathcal{M}, \mu)$ ) the associated quotient space under the relation of equality  $\mu$ -almost everywhere.

We first prove the following lemma.

**Lemma 3.** *Suppose that  $X$  is semireflexive. Let  $\rho$  be a real-valued function, defined on  $X'$ , satisfying:*

- (i)  $\rho(x' + y') \leq \rho(x') + \rho(y')$  and  $\rho(\lambda x') = \lambda \rho(x')$  for  $\lambda \geq 0$ ,
  - (ii) for every  $\epsilon > 0$ ,  $\rho^{-1}((-\infty, \epsilon))$  is a neighbourhood of 0 in  $X'$ .
- Then  $\rho$  is  $\sigma(X', X)$ -lower semicontinuous.

**Proof.** Let  $\alpha \in \mathbb{R}$ ; we prove that the set

$$A = \{x' \in X' : \rho(x') \leq \alpha\}$$

is  $\sigma(X', X)$ -closed. Since  $X$  is semireflexive and  $A$  is convex, it is sufficient to prove that  $A$  is strongly closed. Let  $y' \in \bar{A}$  and let  $\epsilon > 0$ . By (ii), there exists a balanced neighbourhood  $U$  of 0 in  $X'$  such that  $z' \in U$  implies  $\rho(z') < \epsilon$ . Then there is  $x' \in y' + U$  such that  $\rho(x') \leq \alpha$ . It follows that

$$\rho(y') \leq \rho(y' - x') + \rho(x') < \epsilon + \alpha.$$

Therefore  $y' \in A$ , which completes the proof.

**Theorem 2.** *Let  $(S, \mathcal{M}, \mu)$  be a probability space (i.e.  $\mu(S) = 1$ ) and  $X$  a locally convex space that is semireflexive. Assume that  $X'$  contains a sequence which separates the points of  $X$ . Also let  $M$  be a convex compact-valued multimeasure from  $\mathcal{M}$  to  $X$ . Suppose that there exist a convex compact metrizable subset  $K$  of  $X$  and a positive measure  $\nu \ll \mu$  such that for every  $A \in \mathcal{M}$ ,*

$$M(A) \subseteq \nu(A)K.$$

Then there is a convex compact-valued integrable multifunction  $F$  from  $S$  to  $X$  such that

$$M(A) = \int_A F d\mu,$$

for every  $A \in \mathcal{M}$ .

**Proof.** We may suppose that  $K$  is balanced without loss of generality. For every  $x' \in X'$ , we define for each  $A \in \mathcal{M}$ ,

$$\mu_{x'}(A) = \varphi(x', M(A)).$$

Then each  $\mu_{x'}$  is a real-valued bounded measure and these measures satisfy the following properties:

- (i)  $\mu_{x'+y'} \leq \mu_{x'} + \mu_{y'}$ ,
- (ii)  $\mu_{\lambda x'} = \lambda \mu_{x'}$  for  $\lambda \geq 0$ .

Moreover, for each  $x' \in X'$ , we have  $\mu_{x'} \ll \mu$ ; hence there is  $\psi_{x'} \in L^1(S, \mathcal{M}, \mu)$  such that for every  $A \in \mathcal{M}$ ,

$$\mu_{x'}(A) = \int_A \psi_{x'} d\mu.$$

Certainly the functions  $\psi_{x'}$  satisfy the conditions similar to (i) and (ii).

Now we want to find, for each  $x' \in X'$ , a function  $\Psi_{x'}$  in the class  $\psi_{x'}$  such that for every  $t \in S$ , the map  $x' \rightarrow \Psi_{x'}(t)$  satisfies the conditions of Lemma 3. Let  $\theta$  be the density function of  $\nu$  with respect to  $\mu$ . For every  $x' \in X'$  and every  $A \in \mathcal{M}$ , since  $M(A) \subseteq \nu(A)K$ , we have

$$\varphi(x', M(A)) \leq \varphi(x', \nu(A)K) = \nu(A)\varphi(x', K).$$

Hence, putting  $k_{x'} = |\varphi(x', K)|$ , we have  $|\mu_{x'}|(A) \leq \nu(A)k_{x'}$ , for every  $A \in \mathcal{M}$ . Therefore, for every  $x' \in X'$ ,

$$|\psi_{x'}| \leq k_{x'}\theta.$$

Let us choose a non-negative member  $\Theta$  in the class  $\theta$  and put, for  $n = 0, 1, 2, \dots$ ,

$$S_n = \{t \in S : n \leq \Theta(t) < n + 1\}.$$

Thus each  $S_n \in \mathcal{M}$  and the  $S_n$  form a partition for  $S$ . For each  $n = 0, 1, 2, \dots$ , let  $\psi_{x',n}$  be the restriction of  $\psi_{x'}$  on  $S_n$  and define  $\mathcal{M}_n, \mu_n$  analogously. Then  $\psi_{x',n} \in L^\infty(S_n, \mathcal{M}_n, \mu_n)$ . Therefore, by the Lifting Theorem (8, Théorème 12, p. 195), each  $\psi_{x',n}$  can be lifted to a function  $\Psi_{x',n} \in \mathcal{L}^\infty(S_n, \mathcal{M}_n)$  (note that the lifting map is linear, positive and isometric). We obtain the function  $\Psi_{x'}$  by gluing the functions  $\Psi_{x',n}$  together. It is clear that  $\Psi_{x'} \in \mathcal{L}^1(S, \mathcal{M}, \mu)$  and

(iii)  $\Psi_{x'+y'} \leq \Psi_{x'} + \Psi_{y'}$ ,

(iv)  $\Psi_{\lambda x'} = \lambda \Psi_{x'}$  for  $\lambda \geq 0$ .

Now, let  $t$  be chosen and fixed in  $S$ ; then  $t \in S_n$  for some  $n = 0, 1, 2, \dots$ . For every  $x' \in X'$ , since  $\|\psi_{x',n}\|_\infty \leq k_{x'}(n + 1)$ , we have  $\|\Psi_{x',n}\| \leq k_{x'}(n + 1)$  and hence

$$|\Psi_{x'}(t)| \leq k_{x'}(n + 1).$$

According to Hörmander (5, Théorème 7), the function  $x' \rightarrow k_{x'}$  is (strongly) continuous. Hence the function  $x' \rightarrow \Psi_{x'}(t)$  is continuous at 0. This fact, combined with (iii) and (iv), implies that the function  $x' \rightarrow \Psi_{x'}(t)$  is  $\sigma(X', X)$ -lower semicontinuous (Lemma 3). Therefore (by Théorème 5 of (5)), there is a convex closed subset  $F(t)$  of  $X$  such that

$$\Psi_{x'}(t) = \varphi(x', F(t)),$$

for every  $x' \in X'$ . Moreover if  $x' \in K^\circ$ , the polar set of  $K$ , then  $k_{x'} \leq 1$ . It follows that

$$F(t) \subseteq (n + 1)K^\circ = (n + 1)K.$$

Hence  $F(t)$  is compact for each  $t \in S$ .

Next, we prove that the multifunction  $F$  is integrable. Note first that for every  $x' \in X'$ , the function  $t \rightarrow \varphi(x', F(t))$  is measurable and that  $F(t)$  is contained in the convex compact metrizable set  $(n + 1)K$  whenever  $t \in S_n$ . Thus (by Proposition 8 of (14)), the restriction of  $F$  on each  $S_n$  ( $n = 0, 1, 2, \dots$ ) is measurable. Therefore  $F$  is measurable. Then, by Lemma 2,  $F$  is perfectly measurable. Now let  $f \in \mathcal{F}(F)$ ; then for every  $x' \in X'$ ,

$$-\Psi_{-x'} \leq \langle x', f \rangle \leq \Psi_{x'}.$$

This shows that  $f$  is scalarly integrable. Furthermore, for each  $n = 0, 1, 2, \dots$ ,  $f(S_n) \subseteq$

$(n + 1)K$  which is convex compact and balanced. Therefore (by Théorème 1 of (2)),  $f$  is integrable (that is,  $\int_A f d\mu \in X$  for every  $A \in \mathcal{M}$ ). This means that  $F$  is integrable.

Finally, since  $X$  is semireflexive and because every scalarly measurable selector of  $F$  is measurable, we obtain from (2, Théorème 2)

$$\varphi\left(x', \int_A F d\mu\right) = \int_A \varphi(x', F(\cdot)) d\mu,$$

for every  $A \in \mathcal{M}$  and every  $x' \in X'$ . Yet the right-hand side is the same as

$$\int_A \Psi_{x'} d\mu = \int_A \psi_{x'} d\mu = \varphi(x', M(A)).$$

Therefore, by Théorème 1 of (5),

$$M(A) = \int_A F d\mu,$$

for every  $A \in \mathcal{M}$ . This completes the proof.

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