

## Some Properties of Parabolic Curves.

BY GEORGE A. GIBSON, M.A.

If the tangent at a point P on the parabolic curve  $cy = ax^n$  meet the axis of  $x$  at M, it is a well-known property that the area between the radius vector OP and the arc OP is  $n$  times that between the arc OP and the two tangents OM, MP, O being the origin and  $n > 1$ . The converse is also true; for taking any point O on a curve as origin and the tangent at O as axis of  $x$ , let us seek for the locus of P if the area between OP and the arc OP be  $n$  times the area between the arc OP and the tangents OM, MP.

The area between the chord OP and the arc OP is

$$\frac{1}{2}xy - \int_0^x y dx$$

and the area between arc and tangents is

$$\int_0^x y dx - \frac{y^2}{2p}$$

where  $p = dy/dx$ . Hence

$$\frac{1}{2}xy - \int_0^x y dx = n \int_0^x y dx - \frac{ny^2}{2p}$$

Differentiating with respect to  $x$ , the differential equation of the curve will be

$$\frac{ny^2}{p^2} \frac{dp}{dx} = xp - y$$

This may be written

$$n \frac{d}{dx} \left( \frac{1}{p} \right) = \frac{d}{dx} \left( \frac{x}{y} \right)$$

$$\therefore \frac{n}{p} = \frac{x}{y} + C$$

$$\text{i.e. } \frac{dx}{dy} - \frac{1}{ny} x = \frac{C}{n}$$

the integral of which is

$$x = Dy^{\frac{1}{n}} + \frac{C}{n-1}y$$

or

$$y = (Ax + By)^n$$

If  $B=0$ , we have the form  $cy = x^n$ ; and in general if  $Ax + By = 0$  be taken as axis of  $y$  in a system of oblique coordinates, the equation takes the same form  $cy = x^n$ .

If  $n$  were a positive proper fraction, the axes would simply be interchanged.

Consider more particularly the parabola  $x^2 = 4ay$ . In this case the area between  $OP$  and the curve is  $b^3/24a$  if  $b$  is the abscissa of  $P$ , while the area between the arc and the tangents is  $b^3/48a$ . It will be noticed that  $b^3/48a$  is the area between the chord  $OP'$  and the arc  $OP'$  of the parabola  $x^2 + ay = 0$  where  $b/2$  is the abscissa of  $P'$ . But  $b/2$  is the abscissa of  $M$  while the ordinate of  $x^2 + ay = 0$  for the abscissa  $b/2$  is  $-b^2/4a$ , that is, the intercept made by the tangent at  $P$  on the axis of  $y$ . In fact  $x^2 + ay = 0$  is the locus of a point which has for coordinates the intercepts made by the tangent at  $P$  on the axis of  $x$  and  $y$ . (Compare Forsyth's *Diff. Equations*, p. 41 ex. 9.) How far does this property hold for the general parabola? In other words what is the solution of the following problem:—A curve is referred to the tangent and normal at a point  $O$  as axis of  $x$  and  $y$  and the tangent at  $P$  cuts the axis of  $x$  at  $M$  and that of  $y$  at  $N$ ; if the point  $P'$  be taken having  $OM$ ,  $ON$  for coordinates what will be the equations of the loci of  $P$  and of  $P'$  if the area between the chord  $OP'$  and the arc  $OP'$  be  $n$  times the area between the arc  $OP$  and the tangents  $OM$ ,  $MP$ ?

Let  $(x, y)$   $(\xi, \eta)$  be the coordinates of  $P$  and  $P'$  and denote  $dy/dx$  by  $p$ ; then

$$\xi = x - y/p, \quad \eta = y - px.$$

The area between the arc  $OP$  and the tangents  $OM$ ,  $PM$  is

$$\int_0^x xy'd - \frac{y^2}{2p}$$

The area cut off by the chord  $OP'$  from the locus of  $P$  is

$$\int_0^{\xi} \eta d\xi + \frac{y^2}{2p} + \frac{1}{2}px^2 - xy$$

both areas being positive. Hence

$$n \int_0^x y dx - \frac{ny^2}{2p} = \int_0^{\xi} \eta d\xi + \frac{y^2}{2p} + \frac{1}{2}px^2 - xy$$

Differentiating with respect to  $x$  and noting that

$$\frac{d\xi}{dx} = \frac{y}{p^2} \frac{dp}{dx}$$

we get 
$$\left( \frac{n-1}{2} \frac{y^2}{p^2} + \frac{xy}{p} - \frac{1}{2}x^2 \right) \frac{dp}{dx} = 0$$

$dp/dx = 0$  gives no solution. Hence the equation of the locus of  $P$  is given by

$$p^2x^2 - 2xy\mu - (n-1)y^2 = 0$$

or 
$$xp = y(1 \pm \sqrt{n})$$

the integral of which is  $cy = x^1 \pm \sqrt{n}$ , giving only one solution,  $cy = x^2$  when  $n = 1$ .

If  $n$  be not a square each curve is transcendental, but if  $n = m^2$ , we have  $cy = x^{m+1}$  or  $cy = x^{1-m}$ . The solution  $cy = x^{1-m}$  evidently does not satisfy the conditions of the problem, the axis of  $x$  not being the tangent at 0, but obviously the other solution  $cy = x^{m+1}$  does.

To find the locus of  $P'$  we have

$$\xi = \frac{m}{m+1}x, \quad \eta = -\frac{m}{c}x^{m+1}$$

and therefore 
$$\xi^{m+1} + \frac{cm^m}{(m+1)^{m+1}}\eta = 0$$

These are parabolic curves which for  $m = 1$  reduce to the ordinary parabola.

With regard to the solution  $cy = x^{1-m}$ , it may be noted that when  $m$  is greater than two the axes are asymptotes and a similar proposition holds for the two loci. Using the form  $x^{m-1}y = k$  as the equation to the locus of P we find for the locus of P' the equation

$$\xi^{m-1}\eta = \frac{km^n}{(m-1)^{m-1}}$$

The area bounded by the tangent PM, the part of the axis of  $x$  from M to  $+\infty$  and the arc from P to the same end of the axis of  $x$  is

$$\frac{m.v.y}{2(m-1)(m-2)}$$

On the other hand the area bounded by the line OP', the positive part of the axis of  $x$  and the arc of the locus of P' from P' to the positive end of the axis of  $x$  is

$$\frac{m\xi\eta}{2(m-2)} = \frac{m^2xy}{2(m-1)(m-2)}$$

and is therefore  $m^2$ , *i.e.*,  $n$  times the former area.

When  $m$  is less than 1 the tangent at the origin to the curve  $cy = x^{1-m}$  is the axis of  $y$  and a similar proposition to that given for the curve  $cy = x^{m+1}$  holds if M and N be taken on the axes of  $y$  and  $x$  respectively, while if  $m$  be greater than 1 but less than 2 the same change in M, N gives a result analogous to that for the curve  $x^{m-1}y = k$  when  $m$  is greater than 2.

