

# ENDOMORPHISM REGULAR OCKHAM ALGEBRAS OF FINITE BOOLEAN TYPE

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**Abstract.** If  $(L; f)$  is an Ockham algebra with dual space  $(X; g)$ , then it is known that the semigroup of Ockham endomorphisms on  $L$  is (anti-)isomorphic to the semigroup  $\Lambda(X; g)$  of continuous order-preserving mappings on  $X$  that commute with  $g$ . Here we consider the case where  $L$  is a finite boolean lattice and  $f$  is a bijection. We begin by determining the size of  $\Lambda(X; g)$ , and obtain necessary and sufficient conditions for this semigroup to be regular or orthodox. We also describe its structure when it is a group, or an inverse semigroup that is not a group. In the former case it is a cartesian product of cyclic groups and in the latter a cartesian product of cyclic groups each with a zero adjoined.

An *Ockham algebra*  $(L; f)$  is a bounded distributive lattice  $L$  on which there is defined a dual endomorphism  $f$ . For the basic properties of Ockham algebras we refer the reader to [1]. The most obvious example of an Ockham algebra is, of course, a boolean algebra  $(B; ')$ . In general, a boolean lattice can be made into an Ockham algebra in many different ways. Throughout what follows we shall assume that the Ockham algebra  $(L; f)$  is of *finite boolean type*, in the sense that  $L$  is a finite boolean lattice and  $f$  is a dual automorphism. Then  $(L; f)$  necessarily belongs to the Berman class  $\mathbf{K}_{p,0}$  for some  $p$ . If  $L \simeq 2^k$  then, by [1, Chapter 4], the dual space  $(X; g)$  is such that  $X$  is discretely ordered with  $|X| = k$ , and  $g$  is a permutation on  $X$  such that  $g^{2^p} = \text{id}_X$ . We shall denote by  $X_1, \dots, X_m$  the orbits of  $g$ . For each  $i$  we choose and fix a representative  $x_i \in X_i$ . Defining  $c_i = |X_i|$ , for each  $i$ , we therefore have

$$X_i = \{x_i, g(x_i), g^2(x_i), \dots, g^{c_i-1}(x_i)\}.$$

Consider the set  $\Lambda(X; g)$  consisting of those mappings  $\vartheta: X \rightarrow X$  that commute with  $g$ . By duality [2] we know that  $(\Lambda(X; g), \circ)$  is a semigroup that is (anti-)isomorphic to the semigroup  $\text{End}((L; f), \circ)$  of Ockham endomorphisms on  $L$ . By considering  $\Lambda(X; g)$  we can therefore obtain properties of  $\text{End}(L; f)$ . Our principal objective is to determine precisely when this semigroup is regular. For this purpose, we begin by observing the following results.

**THEOREM 1.** *If  $\vartheta \in \Lambda(X; g)$ , then for every  $X_i$  there exists an  $X_j$  such that  $\vartheta(X_i) = X_j$ .*

*Proof.* Given  $x_i$ , let  $j$  be such that  $\vartheta(x_i) \in X_j$ . Observe first that every  $y \in \vartheta(X_i)$  is of the form  $y = \vartheta(g^{r_i}(x_i))$ , where  $0 \leq r_i < c_i$ . Since  $\vartheta, g$  commute, we have  $y = g^{r_i}(\vartheta(x_i)) \in X_j$ . It follows that  $\vartheta(X_i) \subseteq X_j$ . On the other hand, since  $\vartheta(x_i) \in X_j$  we have  $\vartheta(x_i) = g^{s_i}(x_j)$ , where  $0 \leq s_i < c_j$ . It follows that, since  $g^{2^p} = \text{id}_X$ ,

$$x_j = g^{2^p - s_i}(\vartheta(x_i)) = \vartheta(g^{2^p - s_i}(x_i)) \in \vartheta(X_i),$$

whence  $X_j \subseteq \vartheta(X_i)$ . Combining these observations, we obtain  $\vartheta(X_i) = X_j$ .  $\diamond$

**THEOREM 2.** *If  $b_1, \dots, b_m \in X$ , with  $b_i \in X_i$  for each  $i$ , then the following statements are equivalent:*

- (1) *there exists a (necessarily unique)  $\vartheta \in \Lambda(X; g)$  such that  $\vartheta(x_i) = b_i$  for each  $i$ ;*  
 (2)  $c_i \mid c_i$  for each  $i$ .

*Proof.* (1)  $\Rightarrow$  (2). From  $g^{c_i}(x_i) = x_i$  we obtain  $g^{c_i}(\vartheta(x_i)) = \vartheta(x_i)$ . If (1) holds, then  $g^{c_i}(b_i) = b_i$ . Since  $b_j \in X_i$  we deduce that  $c_i \mid c_j$ .

(2)  $\Rightarrow$  (1). Every  $y \in X$  belongs to a unique  $X_i$  and so is of the form  $y = g^r(x_i)$  with  $0 \leq r < c_i$ . We can therefore define mapping  $\vartheta: X \rightarrow X$  by

$$\vartheta(y) = \vartheta(g^r(x_i)) = g^r(b_i).$$

Clearly,  $\vartheta(x_i) = \vartheta(g^0(x_i)) = g^0(b_i) = b_i$ .

To see that  $\vartheta \in \Lambda(X; g)$ , we shall make use of the observation that

$$g^{c_i}(b_i) = b_i. \quad (\text{A})$$

In fact, by (2) we can write  $c_i = d_i c_i$  whence, using the fact that  $g^{c_i}(b_i) = b_i$ , we have  $g^{d_i c_i}(b_i) = b_i$ .

There are two cases to consider. Suppose that  $y = g^r(x_i)$  is an arbitrary element of  $X$ .

- (a) If  $r < c_i - 1$ , then

$$g\vartheta(y) = g(g^r(b_i)) = g^{r+1}(b_i);$$

$$\vartheta g(y) = \vartheta(g^{r+1}(x_i)) = g^{r+1}(b_i).$$

- (b) If  $r = c_i - 1$ , then

$$g\vartheta(y) = g(g^{c_i-1}(b_i)) = g^{c_i}(b_i) = b_i \text{ [by (A)]};$$

$$\vartheta g(y) = \vartheta(g^{c_i-1}(x_i)) = \vartheta(x_i) = b_i.$$

In each case we have  $g\vartheta(y) = \vartheta g(y)$  and so  $\vartheta \in \Lambda(X; g)$ .

That  $\vartheta$  is unique follows from the fact that if  $\varphi \in \Lambda(X; g)$  is also such that  $\varphi(x_i) = b_i$  then, since the  $x_i$  form a set of representatives of the orbits of  $g$ , we have  $\varphi = \vartheta$ .  $\diamond$

**COROLLARY.** For each  $i$ , define  $J_i = \{j; c_j \mid c_i\}$ . Then

$$|\Lambda(X; g)| = \prod_{i=1}^m \left( \sum_{j \in J_i} c_j \right).$$

*Proof.* Let  $B_i = \bigcup_{j \in J_i} X_j$ . Then we can define a mapping  $\psi: \Lambda(X; g) \rightarrow \prod_{i=1}^m B_i$  by the prescription  $\psi(\vartheta) = (\vartheta(x_1), \dots, \vartheta(x_m))$ . It follows immediately from Theorem 2 that  $\psi$  is a bijection. Consequently,

$$|\Lambda(X; g)| = \prod_{i=1}^m |B_i| = \prod_{i=1}^m \left( \sum_{j \in J_i} c_j \right). \quad \diamond$$

In order to investigate the regularity of  $\Lambda(X; g)$  we require the following concepts.

**DEFINITION.** With  $g, m, c_i$  as above, consider the set  $P_g = \{c_i; i = 1, \dots, m\}$  ordered by divisibility. The length of  $(P_g; |)$  will be called the *dimension* of  $g$  and denoted by  $\dim g$ .

**DEFINITION.** For  $i, j, k \in \{1, \dots, m\}$  we shall say that  $(i, j, k)$  is a *pathological triple* associated with  $g$  if  $i, j, k$  are distinct with  $c_k = c_j \mid c_i$  and  $c_j \neq c_i$ .

**THEOREM 3.** *The semigroup  $\Lambda(X; g)$  is regular if and only if  $\dim g < 2$  and there is no pathological triple associated with  $g$ .*

*Proof.*  $\Rightarrow$ . Suppose, by way of obtaining a contradiction, that  $\dim g \geq 2$  or that there is a pathological triple associated with  $g$ . Then there exist distinct  $i, j, k$  such that  $c_k | c_j | c_i$  and  $c_j < c_i$ . By Theorem 2, we can define  $\vartheta \in \Lambda(X; g)$  by setting

$$\vartheta(x_k) = \vartheta(x_j) = x_k, \quad \vartheta(x_i) = x_j, \quad \vartheta(x_t) = x_t (t \notin \{i, j, k\}).$$

Then, by Theorem 1, we have

$$\vartheta(X_k) = \vartheta(X_j) = X_k, \quad \vartheta(X_i) = X_j, \quad \vartheta(X_t) = X_t (t \notin \{i, j, k\}).$$

Since, by hypothesis,  $\Lambda(X; g)$  is regular, there exists  $\varphi \in \Lambda(X; g)$  such that  $\vartheta\varphi\vartheta = \vartheta$ . Applying each side to  $x_i$  we obtain  $\vartheta\varphi(x_j) = x_j$ . It follows from this that  $\varphi(x_j) \in X_i$ . Theorem 2 now gives the contradiction  $c_i | c_j$ .

$\Leftarrow$ . Conversely, suppose that  $\dim g < 2$  and there is no pathological triple associated with  $g$ . Observe first that if  $\vartheta \in \Lambda(X; g)$  then, for every  $i$ ,

$$|\vartheta(X_i)| \neq |X_i| \Rightarrow \vartheta^2(X_i) = \vartheta(X_i). \tag{1}$$

In fact, given  $\vartheta \in \Lambda(X; g)$  let  $i$  be such that  $|\vartheta(X_i)| \neq |X_i|$ . By Theorem 1 there exist  $j, k$  such that  $\vartheta(X_i) = X_j$  and  $\vartheta^2(X_i) = \vartheta(X_j) = X_k$ ; also by Theorem 2, we have  $c_k | c_j | c_i$ . Note that  $j \neq i$  since, by hypothesis,  $c_j \neq c_i$  and consequently, since  $\dim g < 2$ , we have  $c_k = c_j < c_i$ . It follows that  $k \neq i$ . In fact, we have  $k = j$ , for otherwise  $(i, j, k)$  would be a pathological triple, contradicting the hypothesis. Consequently,  $\vartheta^2(X_i) = \vartheta(X_j) = X_k = X_j$  and therefore  $\vartheta^2(X_i) = \vartheta(X_i)$ .

Now let  $\vartheta \in \Lambda(X; g)$  and consider the set

$$A = \{j; x_j \in \vartheta(X)\}.$$

Let  $\pi: A \rightarrow \{1, \dots, m\}$  be given by

$$\pi(j) = \min\{i; x_j \in \vartheta(X_i)\}.$$

Since  $x_j \in \vartheta(X_{\pi(j)})$  it follows from Theorem 1 that  $\vartheta(X_{\pi(j)}) = X_j$ . Consequently, for every  $j \in A$ , there is a unique  $s_j \in \{0, \dots, c_j - 1\}$  such that

$$\vartheta(x_{\pi(j)}) = g^{s_j}(x_j).$$

Define

$$l_j = \begin{cases} 0, & \text{if } s_j = 0; \\ c_j - s_j, & \text{otherwise.} \end{cases}$$

Then from the above we have

$$x_j = \vartheta(g^{l_j}(x_{\pi(j)})). \tag{2}$$

Consider next the sets

$$A_1 = \{j \in A; c_j = c_{\pi(j)}\}, \quad A_2 = A \setminus A_1.$$

Note that for every  $j \in A_2$  there is a unique  $v_j \in \{0, \dots, c_j - 1\}$  such that

$$\vartheta(x_j) = g^{v_j}(x_j). \quad (3)$$

In fact, we have  $|\vartheta(X_{\pi(j)})| = |X_j| = c_j \neq c_{\pi(j)} = |X_{\pi(j)}|$  and so, by (1), we deduce that  $\vartheta^2(X_{\pi(j)}) = \vartheta(X_{\pi(j)})$ ; that is  $\vartheta(X_j) = X_j$ , whence (3) follows.

Using Theorem 2 we can now define  $\varphi \in \Lambda(X; g)$  by setting

$$\varphi(x_j) = \begin{cases} x_j, & \text{if } j \notin A_i, \\ g^{l_j}(x_{\pi(j)}), & \text{if } j \in A_1; \\ g^{v_j(c_j-1)}(x_j), & \text{if } j \in A_2. \end{cases}$$

We show as follows that  $\vartheta\varphi\vartheta = \vartheta$ , whence  $\Lambda(X; g)$  is regular. For this purpose, observe that  $\varphi$  satisfies the property

$$(\forall j \in A) \quad \vartheta\varphi(x_j) = x_j. \quad (4)$$

In fact, if  $j \in A_1$  then, by (2), we have

$$\vartheta\varphi(x_j) = \vartheta(g^{l_j}(x_{\pi(j)})) = x_j;$$

also if  $j \in A_2$  then, by (3),

$$\begin{aligned} \vartheta\varphi(x_j) &= \vartheta(g^{v_j(c_j-1)}(x_j)) = g^{v_j(c_j-1)}(\vartheta(x_j)) \\ &= g^{v_j(c_j-1)}(g^{v_j}(x_j)) \\ &= g^{v_j c_j}(x_j) \\ &= \underbrace{g^{c_j} \dots g^{c_j}}_{v_j}(x_j) = x_j. \end{aligned}$$

Now let  $x \in X$  with  $\vartheta(x) \in X_j$ . Then  $\vartheta(x) = g^r(x_j)$  for some  $r$  with  $0 \leq r \leq c_j - 1$ , whence  $g^{2p-r}(\vartheta(x)) = \vartheta(g^{2p-r}(x)) = x_j$ . It follows that  $j \in A$ . Applying  $\vartheta\varphi$ , we deduce by (4) that

$$g^{2p-r}\vartheta\varphi\vartheta(x) = \vartheta\varphi(x_j) = x_j = g^{2p-r}\vartheta(x),$$

whence,  $g$  being a bijection,  $\vartheta\varphi\vartheta(x) = \vartheta(x)$  and so  $\vartheta\varphi\vartheta = \vartheta$ , as required.  $\diamond$

**EXAMPLE 1.** By Theorem 3, the smallest  $X$  such that  $\Lambda(X; g)$  is not regular arises when  $m = 3$  and

$$X_1 = \{x_1\}, \quad X_2 = \{x_2\}, \quad X_3 = \{x_3, g(x_3)\}.$$

Hence  $|X| = 4$  with  $c_1 = 1$ ,  $c_2 = 1$ , and  $c_3 = 2$ . By the Corollary of Theorem 2,  $|\Lambda(X; g)| = 2 \cdot 2 \cdot 4 = 16$ .

We now proceed to consider the question of when, as a regular semigroup,  $\Lambda(X; g)$

is orthodox, inverse, or a group. We begin with the orthodox case, and for this purpose we shall use the following results.

**THEOREM 4.** *If  $c_i = c_j > 1$  for some  $i, j$  with  $i \neq j$ , then  $\Lambda(X; g)$  is not orthodox.*

*Proof.* Without loss of generality, we may suppose that  $c_1 = c_2 > 1$ . Using Theorem 2, we can define  $\alpha, \beta \in \Lambda(X; g)$  as follows:

$$\begin{aligned} \alpha(x_1) &= x_2, & \alpha(x_i) &= x_i (i \geq 2); \\ \beta(x_1) &= x_1, & \beta(x_2) &= g(x_1), & \beta(x_i) &= x_i (i \geq 3). \end{aligned}$$

Clearly,  $\alpha$  and  $\beta$  are idempotent, but  $\alpha\beta$  is not. To see this, observe that

$$\begin{aligned} \alpha\beta(x_2) &= \alpha g(x_1) = g\alpha(x_1) = g(x_2); \\ \alpha\beta\alpha\beta(x_2) &= \alpha\beta g(x_2) = g\alpha\beta(x_2) = g^2(x_2). \end{aligned}$$

Since  $c_2 > 1$  we have  $g^2(x_2) \neq g(x_2)$  and therefore  $\alpha\beta$  is not idempotent.  $\diamond$

**THEOREM 5.** *If  $|X| > 2$  and  $c_i = c_j = 1$  for some  $i, j$  with  $i \neq j$ , then  $\Lambda(X; g)$  is not orthodox.*

*Proof.* Since  $|X| > 2$  there are at least three distinct orbits. Without loss of generality we may assume that  $c_1 = c_2 = 1$ . Using Theorem 2, we can define  $\alpha, \beta \in \Lambda(X; g)$  as follows:

$$\begin{aligned} \alpha(x_1) &= x_1, & \alpha(x_2) &= x_2, & \alpha(x_3) &= x_1, & \alpha(x_i) &= x_i, (i \geq 4); \\ \beta(x_1) &= x_2, & \beta(x_i) &= x_i, (i \geq 2). \end{aligned}$$

Clearly,  $\alpha$  and  $\beta$  are idempotent. However

$$\begin{aligned} \alpha\beta(x_3) &= \alpha(x_3) = x_1, \\ \alpha\beta\alpha\beta(x_3) &= \alpha\beta(x_1) = \alpha(x_2) = x_2, \end{aligned}$$

and so  $\alpha\beta$  is not idempotent.  $\diamond$

If  $M_g$  denotes the set of minimal elements of the ordered set  $(P_g, |)$ , let

$$\Delta_g = \{i; c_i \in M_g\}.$$

For each  $\vartheta \in \Lambda(X; g)$  define also

$$I_\vartheta = \{i; \vartheta(X_i) = X_i\}.$$

**THEOREM 6.** *If the  $c_i$  are distinct, then*

$$\Delta_g = \bigcap_{\vartheta} I_\vartheta.$$

*Proof.* Let  $i \in \Delta_g$  and  $\vartheta \in \Lambda(X; g)$ . By Theorem 1, we have  $\vartheta(X_i) = X_j$  for some  $j$ ; also, by Theorem 2, we have  $c_j | c_i$ . Since  $c_i$  is minimal in  $P_g$ , it follows that  $c_j = c_i$ , whence  $j = i$ . Thus  $\vartheta(X_i) = X_i$  and consequently  $i \in I_\vartheta$ . Hence we see that  $\Delta_g \subseteq \bigcap_{\vartheta} I_\vartheta$ .

To obtain the reverse inclusion, let  $\Delta'_g$  denote the complement of  $\Delta_g$  in  $\{1, \dots, m\}$  and observe from the definitions that

$$(\forall i \in \Delta'_g)(\exists j \in \Delta_g) c_j | c_i.$$

There therefore exists a mapping  $\pi: \Delta'_g \rightarrow \Delta_g$  with the property that  $c_{\pi(i)} | c_i$ . Consequently, by Theorem 2, we can define  $\varphi \in \Lambda(X; g)$  by setting

$$\varphi(x_i) = \begin{cases} x_i, & \text{if } i \in \Delta_g; \\ x_{\pi(i)}, & \text{if } i \notin \Delta_g. \end{cases}$$

It suffices to prove that  $I_\varphi \subseteq \Delta_g$ . Suppose, by way of obtaining a contradiction, that  $i \in I_\varphi$  and  $i \notin \Delta_g$ . Then since  $i \in I_\varphi$  we have  $\varphi(X_i) = X_i$ ; also, since  $i \notin \Delta_g$ , we have  $\varphi(x_i) = x_{\pi(i)}$  whence, by Theorem 1,  $\varphi(X_i) = X_{\pi(i)}$ . We conclude that  $X_i = X_{\pi(i)}$ , whence  $i = \pi(i)$ , which is absurd since by hypothesis  $i \notin \Delta_g$  and  $\pi(i) \in \Delta_g$ .  $\diamond$

**THEOREM 7.** *Suppose that  $\dim g < 2$  and the  $c_i$  are distinct. If  $\vartheta(X_i) \neq X_i$ , then  $\vartheta(X_i) = X_j$  for some  $j \in \Delta_g$ .*

*Proof.* By Theorem 1, there exists  $j$  such that  $\vartheta(X_i) = X_j$  with clearly,  $j \neq i$ . Also, by Theorem 2, we have  $c_j | c_i$  with  $c_j < c_i$ . The hypothesis that  $\dim g < 2$  now gives  $c_j \in M_g$  and therefore  $j \in \Delta_g$ .  $\diamond$

**THEOREM 8.** *If  $\dim g < 2$  and the  $c_i$  are distinct, then*

$$I_\vartheta \cap I_\varphi = I_{\vartheta\varphi} \quad (\forall \vartheta, \varphi \in \Lambda(X; g))$$

*Proof.* If  $i \in I_\vartheta \cap I_\varphi$ , then  $\vartheta\varphi(X_i) = \vartheta(X_i) = X_i$  gives  $i \in I_{\vartheta\varphi}$ . Conversely, if  $i \in I_{\vartheta\varphi}$ , then we have

$$\vartheta\varphi(X_i) = X_i. \tag{B}$$

Suppose, by way of obtaining a contradiction that  $i \notin I_\varphi$ . Then  $\varphi(X_i) \neq X_i$  and so, by Theorem 7, there exists  $j \in \Delta_g$  such that  $\varphi(X_i) = X_j$ . By Theorem 6, we have  $j \in I_\vartheta$  and so  $\vartheta(X_j) = X_j$ . Consequently,  $\vartheta\varphi(X_i) = \vartheta(X_j) = X_j$ , which contradicts (B) since  $i \neq j$ . Hence  $i \in I_\varphi$  and so  $\varphi(X_i) = X_i$ . It now follows from (B) that  $\vartheta(X_i) = X_i$  and so  $i \in I_\vartheta$ . Hence  $i \in I_\vartheta \cap I_\varphi$ .  $\diamond$

In order to determine when  $\Lambda(X; g)$  is orthodox we must consider separately the cases  $|X| \leq 2$  and  $|X| > 2$ . In the former case,  $\Lambda(X; g)$  is always orthodox. In fact,  $\Lambda(X; g)$  reduces to the trivial group when  $|X| = 1$ , and when  $|X| = 2$  there are two possibilities.

(1) *g has a single orbit (that is,  $m = 1$ ).*

In this case  $\Lambda(X; g)$  is isomorphic to the group  $\mathbb{Z}_2$ .

(2) *g has two orbits (that is,  $m = 2$ ).*

In this case,  $g$  is the identity map on  $X$  and  $\Lambda(X; g)$  is the full transformation semigroup on  $X$ , and for  $|X| = 2$  this is orthodox.

The situation when  $|X| > 2$  is as follows.

**THEOREM 9.** *If  $|X| > 2$ , then the semigroup  $\Lambda(X; g)$  is orthodox if and only if  $\dim g < 2$  and the  $c_i$  are distinct.*

*Proof.*  $\Rightarrow$ . Suppose that  $\Lambda(X; g)$  is orthodox. Then, by Theorem 3,  $\dim g < 2$  and, by Theorems 4 and 5, the  $c_i$  are distinct.

⇐. Conversely, suppose that the conditions hold. Since the  $c_i$  are distinct, there can be no pathological triple associated with  $g$  and therefore, by Theorem 3,  $\Lambda(X; g)$  is regular. To see that it is orthodox, let  $\vartheta, \varphi$  be idempotents of  $\Lambda(X; g)$ . Then we have

$$\vartheta\varphi(X) \subseteq \varphi(X). \tag{C}$$

To see this, observe that

(1) if  $i \in I_{\vartheta\varphi}$ , then  $\vartheta\varphi(X_i) = X_i$  and so, by Theorem 8,  $\vartheta(X_i) = X_i$  and  $\varphi(X_i) = X_i$ , whence  $\vartheta\varphi(X_i) = \varphi(X_i) \subseteq \varphi(X)$ ;

(2) If  $i \notin I_{\vartheta\varphi}$ , then by Theorem 7 there exists  $j \in \Delta_g$  such that  $\vartheta\varphi(X_i) = X_j$ . But, by Theorem 6,  $\varphi(X_j) = X_j$ . Hence we have  $\vartheta\varphi(X_i) = \varphi(X_j) \subseteq \varphi(X)$ .

Thus, in all cases,  $\vartheta\varphi(X_i) \subseteq \varphi(X)$  and therefore  $\vartheta\varphi(X) \subseteq \varphi(X)$ .

Using (C) we now have, for every  $x \in X$ ,  $\vartheta\varphi(x) \in \varphi(X)$  and so there exists  $t \in X$  such that  $\vartheta\varphi(x) = \varphi(t)$ . Consequently,

$$\varphi\vartheta\varphi(x) = \varphi^2(t) = \varphi(t) = \vartheta\varphi(x)$$

and so  $\varphi\vartheta\varphi = \vartheta\varphi$ . It follows from this that  $\vartheta\varphi$  is idempotent, whence  $\Lambda(X; g)$  is orthodox.  $\diamond$

EXAMPLE 2. The smallest  $X$  such that  $\Lambda(X; g)$  is regular but not orthodox arises when  $m = 3$  and  $c_1 = c_2 = c_3 = 1$ . In this case we have  $|X| = 3$  and  $|\Lambda(X; g)| = 3 \cdot 3 \cdot 3 = 27$ .

We now proceed to consider the question of when  $\Lambda(X; g)$  is a group and, more generally, an inverse semigroup. As we shall see, in each of these cases it is commutative. For this purpose we shall use the following results.

THEOREM 10. *If  $c_i = c_j$ , for some  $i, j$  with  $i \neq j$ , then  $\Lambda(X; g)$  contains a pair of idempotents that do not commute.*

*Proof.* Suppose, without loss of generality, that  $c_1 = c_2$ . Using Theorem 2 we can define  $\alpha, \beta \in \Lambda(X; g)$  as follows:

$$\begin{aligned} \alpha(x_1) = \alpha(x_2) = x_1, & \quad \alpha(x_i) = x_i \ (i \geq 3); \\ \beta(x_1) = \beta(x_2) = x_2, & \quad \beta(x_i) = x_i \ (i \geq 3). \end{aligned}$$

Clearly,  $\alpha$  and  $\beta$  are idempotent. Moreover,  $\alpha\beta(x_1) = x_1$  and  $\beta\alpha(x_1) = x_2$ , so that  $\alpha\beta \neq \beta\alpha$ .  $\diamond$

THEOREM 11. *If  $1 \neq c_i | c_j$ , for some  $i, j$  with  $i \neq j$ , then  $\Lambda(X; g)$  contains a pair of idempotents that do not commute.*

*Proof.* Suppose, without loss of generality, that  $1 \neq c_1 | c_2$ . Using Theorem 2 we can define  $\alpha, \beta \in \Lambda(X; g)$  as follows;

$$\begin{aligned} \alpha(x_1) = \alpha(x_2) = x_1, & \quad \alpha(x_i) = x_i \ (i \geq 3); \\ \beta(x_1) = x_1, & \quad \beta(x_2) = g(x_1), & \quad \beta(x_i) = x_i \ (i \geq 3). \end{aligned}$$

Clearly,  $\alpha$  and  $\beta$  are idempotent. Moreover,  $\alpha\beta(x_2) = g(x_1)$  and  $\beta\alpha(x_2) = x_1$ . Since  $c_1 \neq 1$  it follows that  $\alpha\beta \neq \beta\alpha$ .  $\diamond$

THEOREM 12. *The following statements are equivalent:*

- (1)  $\Lambda(X; g)$  is a group;

(2)  $\dim g = 0$  and the  $c_i$  are distinct.

Moreover, as a group,  $\Lambda(X; g) \simeq \prod_{i=1}^m \mathbb{Z}_{c_i}$ .

*Proof.* (1)  $\Rightarrow$  (2). Suppose that  $\Lambda(X; g)$  is a group. If  $m = 1$ , then  $X = X_1$  and so  $\dim g = 0$ . Also, by Theorem 2, every  $\vartheta \in \Lambda(X; g)$  is given by  $\vartheta(x_1) = g^r(x_1)$ , where  $0 \leq r < |X|$ . Since  $g$  has a single orbit it follows that  $\vartheta = g^r$  and therefore  $\Lambda(X; g) = \mathbb{Z}_{|X|}$ .

If now  $m > 1$  then, by Theorem 10, the  $c_i$  are distinct. Moreover,  $c_i \neq 1$  for every  $i$ . To see this, suppose that (say)  $c_1 = 1$ . Then, applying Theorem 2 with  $b_1 = b_2 = x_1$  and  $b_i = x_i$  for  $i \geq 3$ , we obtain  $\vartheta \in \Lambda(X; g)$  such that

$$\vartheta(x_1) = \vartheta(x_2) = x_1, \quad \vartheta(x_i) = x_i, \quad (i \geq 3).$$

This is not possible since by hypothesis  $\Lambda(X; g)$  is a group and  $\vartheta$  is not a bijection.

Suppose now, by way of obtaining a contradiction, that  $\dim g \geq 1$ . Then there exist  $c_i, c_j$  with  $c_i \mid c_j$  and  $c_i < c_j$ . Since  $c_i \neq 1$  we can apply Theorem 11 to produce a pair of non-commuting idempotents in  $\Lambda(X; g)$ . This contradicts the hypothesis that  $\Lambda(X; g)$  is a group, and so we conclude that  $\dim g = 0$ .

(2)  $\Rightarrow$  (1). Conversely, suppose that  $\dim g = 0$  and that the  $c_i$  are distinct. Then for every  $\vartheta \in \Lambda(X; g)$  we have, for all  $i$ ,

$$\vartheta(X_i) = X_i. \tag{D}$$

In fact, by Theorem 1, for each  $i$  there exists  $j$  such that  $\vartheta(X_i) = X_j$ . If  $j \neq i$ , then by Theorem 2 we have  $c_j \mid c_i$ , whence the contradiction  $\dim g \geq 1$ . Thus we must have  $j = i$  and so  $\vartheta(X_i) = X_i$ .

It follows from (D) that  $\vartheta$  is surjective, and therefore bijective, since  $X$  is finite. Hence  $\Lambda(X; g)$  is a group.

As for the final statement, observe by (D) that every  $\vartheta \in \Lambda(X; g)$  induces a mapping  $\vartheta_i: X_i \rightarrow X_i$  given by  $\vartheta_i(x) = \vartheta(x)$ . Let  $g_i$  be the bijection induced on  $X_i$  by  $g$  and consider the mapping

$$\psi: \Lambda(X; g) \rightarrow \prod_{i=1}^m \Lambda(X_i; g_i)$$

given by the prescription  $\psi(\vartheta) = (\vartheta_1, \dots, \vartheta_m)$ .

To see that  $\psi$  is surjective observe that if we choose  $\varphi_1, \dots, \varphi_m$  with  $\varphi_i \in \Lambda(X_i; g_i)$  for each  $i$ , then we can define  $\varphi \in \Lambda(X; g)$  by setting  $\varphi(x) = \varphi_i(x)$  when  $x \in X_i$ , thereby obtaining  $\psi(\varphi) = (\varphi_1, \dots, \varphi_m)$ . That  $\psi$  is injective is immediate from the fact that if  $\vartheta, \varphi$  induce the same mapping on the orbits then they are equal. Thus  $\psi$  is a bijection. Since clearly  $(\vartheta\varphi)_i = \vartheta_i\varphi_i$ , it follows that  $\psi$  is an isomorphism.

Finally, since in the Ockham space  $(X_i; g_i)$  the bijection  $g_i$  has only one orbit we have, from the above,  $\Lambda(X_i; g_i) \simeq \mathbb{Z}_{|X_i|} = \mathbb{Z}_{c_i}$ . We conclude that  $\Lambda(X; g) \simeq \prod_{i=1}^m \mathbb{Z}_{c_i}$ .  $\diamond$

COROLLARY 1. *If  $\Lambda(X; g)$  is a group, then*

$$|\Lambda(X; g)| = \prod_{i=1}^m c_i. \quad \diamond$$

COROLLARY 2. For every integer  $k \geq 2$  there exists an Ockham algebra  $(L; f)$  of finite boolean type such that  $(L; f) \in \mathbf{P}_{k,0}$  and  $\text{End}(L; f)$  is a group isomorphic to  $\mathbb{Z}_k$ .

*Proof.* Consider the prime factorisation  $k = \prod_{i=1}^m p_i^{\alpha_i}$ . For  $i = 1, \dots, m$  let  $X_i = \{x_{i,1}, \dots, x_{i,c_i}\}$  be a set of cardinality  $c_i = p_i^{\alpha_i}$  and choose the  $X_i$  to be pairwise disjoint. Let  $X = \bigcup_{i=1}^m X_i$  and define  $g: X \rightarrow X$  by

$$g(x_{i,j}) = x_{i,j+1} \pmod{c_i}.$$

Then clearly  $g$  is a permutation on  $X$  with orbits  $X_1, \dots, X_m$ . Now since  $k = \text{lcm}\{c_i; i = 1, \dots, m\}$  we have  $g^k = \text{id}_X$  and therefore the Ockham space  $(X; g)$  is such that its dual algebra  $(L; f)$  is of finite boolean type and belongs to the class  $\mathbf{P}_{k,0}$ . Since clearly  $\dim g = 0$  and the  $c_i$  are distinct, it follows from Theorem 12 that  $\Lambda(X; g)$  is a group, of the same cardinality as the group of Ockham automorphisms on  $L$ . For each  $i$  we have

$$J_i = \{j; c_j \mid c_i\} = \{j; p_j^{\alpha_j} \mid p_i^{\alpha_i}\} = \{i\}.$$

Consequently, by the Corollary of Theorem 2,

$$|\Lambda(X; g)| = \prod_{i=1}^m \sum_{j \in J_i} c_j = \prod_{i=1}^m c_i = \prod_{i=1}^m p_i^{\alpha_i} = k,$$

and, by Theorem 12,  $\Lambda(X; g) \simeq \mathbb{Z}_k$ .  $\diamond$

THEOREM 13. The following statements are equivalent:

- (1)  $\Lambda(X; g)$  is an inverse semigroup that is not a group;
- (2)  $\dim g = 1$ , the  $c_i$  are distinct, and  $g$  has a (necessarily unique) fixed point.

*Proof.* (1)  $\Rightarrow$  (2). If (1) holds then, by Theorem 3,  $\dim g < 2$  and, by Theorem 10, the  $c_i$  are distinct. Since  $\Lambda(X; g)$  is not a group, it follows by Theorem 12 that  $\dim g = 1$ .

To prove that  $g$  has a fixed point, we show that at least one  $c_i = 1$  (whence only one  $c_i = 1$ ). Suppose, to the contrary, that every  $c_i \neq 1$ . Since  $\dim g = 1$  there exist  $c_i, c_j$  with  $c_i \mid c_j$ . It follows from Theorem 11 that  $\Lambda(X; g)$  is not an inverse semigroup, a contradiction. Hence  $g$  has a fixed point.

(2)  $\Rightarrow$  (1). If (2) holds then, by Theorem 3,  $\Lambda(X; g)$  is regular and, by Theorem 12, it is not a group. Let  $x_1$  be the fixed point of  $g$ . Then for every  $\vartheta \in \Lambda(X; g)$  we have

$$\vartheta(X_i) = X_i \quad \text{or} \quad \vartheta(X_i) = X_1 = \{x_1\}. \tag{E}$$

In fact, by Theorem 1 we have  $\vartheta(X_i) = X_j$  for some  $j$ . Suppose that  $j \neq i$ . Then, by Theorem 2,  $c_j \mid c_i$ . We must then have  $c_j = 1$  since otherwise  $\dim g \geq 2$ , a contradiction. Hence  $X_j = X_1$  and consequently we have that  $\vartheta(X_i) = X_1$ .

We now use (E) to prove that  $\Lambda(X; g)$  is inverse. For this purpose, let  $\alpha, \beta \in \Lambda(X; g)$ . We shall show that  $\alpha, \beta$  commute, and for this it suffices to prove that  $\alpha\beta$  and  $\beta\alpha$  coincide on every  $X_i$ .

Since  $x_1$  is the only fixed point of  $g$ , it is clear that  $\alpha\beta$  and  $\beta\alpha$  agree on  $X_1 = \{x_1\}$ . To show that they agree on  $X_i$  with  $i \neq 1$ , we must consider several cases.

- (a)  $\beta(X_i) = X_i$ .

In this case  $\beta(x_i) = g^r(x_i)$ , where  $0 \leq r < c_i$  and  $\alpha\beta(x_i) = g^r(\alpha(x_i))$ . Two sub-cases, according to (E), arise:

$$(a_1) \alpha(X_i) = X_i.$$

Here  $\alpha(x_i) = g^s(x_i)$  where  $0 \leq s < c_i$ , and we have

$$\begin{aligned} \alpha\beta(x_i) &= g^{r+s}(x_i); \\ \beta\alpha(x_i) &= \beta(g^s(x_i)) = g^s(\beta(x_i)) = g^{r+s}(x_i). \end{aligned}$$

$$(a_2) \alpha(X_i) = X_1.$$

Here we have

$$\begin{aligned} \alpha\beta(x_i) &= g^r(\alpha(x_i)) = g^r(x_1) = x_1; \\ \beta\alpha(x_i) &= \beta(x_1) = x_1. \end{aligned}$$

$$(b) \beta(X_i) = X_1.$$

In this case similar calculations to the above reveal that  $\alpha\beta(x_i) = \beta\alpha(x_i)$ .

It follows from the above that  $\alpha\beta = \beta\alpha$ . Hence  $\Lambda(X; g)$  is a commutative regular semigroup and therefore is inverse.  $\diamond$

**COROLLARY.** *If  $\Lambda(X; g)$  is inverse, then it is commutative.*

*Proof.* This follows from the above and Theorem 12.  $\diamond$

**EXAMPLE 3.** By Theorem 13, the smallest  $X$  such that  $\Lambda(X; g)$  is inverse but not a group arises when  $m = 2$  and  $c_1 = 1, c_2 = 2$ . Here  $|X| = 3$  and  $|\Lambda(X; g)| = 3$ . The smallest corresponding algebra is such that  $L \approx 2^3$ .

**EXAMPLE 4.** The smallest  $X$  such that  $\Lambda(X; g)$  is orthodox but not inverse arises when  $m = 2$  and  $c_1 = 2, c_2 = 4$ . Here  $|X| = 6$  and  $|\Lambda(X; g)| = 2(2 + 4) = 12$ .

When  $\Lambda(X; g)$  is inverse but not a group, we can describe its structure in terms of a cartesian product of cyclic groups each with a zero adjoined.

**THEOREM 14.** *If  $\Lambda(X; g)$  is inverse but not a group, then*

$$\Lambda(X; g) \approx \prod_{i=2}^m \mathbb{Z}_{c_i}^0.$$

*Proof.* As before, let  $x_1$  be the fixed point of  $g$ . For each  $i \geq 2$  define  $Y_i = X_i \cup \{x_1\}$ . Define  $g_i: Y_i \rightarrow Y_i$  by  $g_i(y) = g(y)$  and let  $\eta_i: Y_i \rightarrow Y_i$  be the constant mapping given by  $\eta_i(y) = x_1$ . Then  $\Lambda(Y_i; g_i)$  is a group with zero. To see this, let  $\alpha \in \Lambda(Y_i; g_i)$ . Then either  $\alpha(x_i) = x_1$  (in which case  $\alpha = \eta_i$ ) or  $\alpha(x_i) \in X_i$  and so  $\alpha(x_i) = g_i^r(x_i)$  with  $0 \leq r < c_i$  (in which case  $\alpha = g_i^r$  and is therefore a bijection). It follows that  $\Lambda(Y_i; g_i) \approx \mathbb{Z}_{c_i}^0$ .

Now define  $\psi: \Lambda(X; g) \rightarrow \prod_{i=2}^m \Lambda(Y_i; g_i)$  as in the proof of Theorem 12 to obtain an isomorphism  $\Lambda(X; g) \approx \prod_{i=2}^m \mathbb{Z}_{c_i}^0$ .  $\diamond$

**COROLLARY 1.** *If  $\Lambda(X; g)$  is inverse but not a group, then*

$$|\Lambda(X; g)| = \prod_{i=2}^m (1 + c_i). \quad \diamond$$

**COROLLARY 2.** *For every integer  $k \geq 2$  there exists an Ockham algebra  $(L; f)$  of finite boolean type such that  $(L; f) \in \mathbf{P}_{k,0}$  and  $\text{End}(L; f)$  is an inverse semigroup isomorphic to  $\mathbb{Z}_k^0$ .*

*Proof.* Let  $X_1 = \{x_{1,1}\}$  and  $X_2 = \{x_{2,1}, \dots, x_{2,k}\}$  be disjoint sets of cardinalities 1 and  $k$ . Define a permutation  $g$  on  $X = X_1 \cup X_2$  by setting

$$g(x_{1,1}) = x_{1,1}, \quad g(x_{2,i}) = x_{2,i+1} \pmod{k}.$$

Clearly,  $X_1, X_2$  are the orbits of  $g$ . By Theorems 13 and 14 we have that  $\Lambda(X; g)$  is an inverse semigroup isomorphic to  $\mathbb{Z}_k^0$ .  $\diamond$

Finally, in view of the above results, it is natural to consider the situation that holds when  $|\Lambda(X; g)|$  is a prime.

**THEOREM 15.** *If  $(L; f)$  is an endomorphism regular Ockham algebra of finite boolean type and  $\text{End}(L; f)$  is of prime cardinality  $p$  then  $\text{End}(L; f)$  is isomorphic either to the group  $\mathbb{Z}_p$  or to the inverse semigroup  $\mathbb{Z}_{p-1}^0$  ( $p \geq 3$ ).*

*Proof.* As usual, let  $(X; g)$  be the dual space of  $(L; f)$ . If  $\Lambda(X; g)$  is a group then, by Theorem 12 and its Corollary 1, we have necessarily  $m = 1$  and  $c_1 = p$ , whence  $\Lambda(X; g) \cong \mathbb{Z}_p$ .

Suppose now that  $\Lambda(X; g)$  is not a group. Without loss of generality we may suppose that  $c_1 \leq c_2 \leq \dots \leq c_m$ . Observe that

(a)  $p \geq 3$ .

In fact, suppose that  $p = 2$ . Then necessarily  $g \neq \text{id}_X$ ; for otherwise  $\Lambda(X; g)$  is the full transformation semigroup on  $X$ , and this can never have cardinality 2. Since, by hypothesis,  $|\Lambda(X; g)| = 2$  it follows that  $\Lambda(X; g) = \{\text{id}_X, g\} \cong \mathbb{Z}_2$ , a contradiction.

(b)  $m \geq 2$ .

In fact, suppose that  $m = 1$ . Then, by the Corollary of Theorem 2, we have  $c_1 = p$  whence  $\Lambda(X; g)$  is a group by Theorem 12, a contradiction.

(c)  $c_1 < c_2$ .

In fact, if  $c_1 = c_2$  then  $J_1 = \{j; c_j | c_1\} = \{j; c_j | c_2\} = J_2$ , whence we have  $\sum_{j \in J_1} c_j = \sum_{j \in J_2} c_j = r$  say, and clearly  $r \geq 2$ . The Corollary of Theorem 2 now gives the contradiction  $r^2 | p$ .

(d)  $\sum_{j \in J_1} c_j = 1$ .

In fact, if  $\sum_{j \in J_1} c_j \neq 1$  then by the Corollary of Theorem 2 and the fact that  $p$  is prime we have  $\prod_{i=2}^m (\sum_{j \in J_i} c_j) = 1$ , whence  $c_1 < c_2 = \dots = c_m = 1$  which is absurd.

It follows from these observations that  $J_1 = \{1\}$  and  $c_1 = 1$ . Then  $\prod_{i=2}^m (\sum_{j \in J_i} c_j) = p$  gives  $m = 2$  and  $\sum_{j \in J_2} c_j = p$ . It follows that  $J_2 = \{1, 2\}$  and therefore  $p = c_1(c_1 + c_2) = 1 + c_2$  whence  $c_2 = p - 1$ . It now follows by Theorems 13 and 14 that  $\Lambda(X; g) \cong \mathbb{Z}_{p-1}^0$ .  $\diamond$

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