NORMAL CURVATURE OF MINIMAL SUBMANIFOLDS IN A SPHERE

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1. Introduction. Simons [**5**] has proved a pinching theorem for compact minimal submanifolds in a unit sphere, which led to an intrinsic rigidity result. Sakaki [**4**] improved this result of Simons for arbitrary codimension and has proved that if the scalar curvature S of the minimal submanifold M^n of S^{n+p} satisfies

$$\frac{n(n-1)(2n^2+n-8)}{2(n^2+n-3)} \leq S$$

then either M^n is totally geodesic or $S = 2/3$ in which case $n = 2$ and M^2 is the Veronese surface in a totally geodesic 4-sphere. This result of Sakaki was further improved by Shen [**6**] but only for dimension $n = 3$, where it is shown that if $S > 4$, then M^3 is totally geodesic (cf. Theorem 3, p. 791).

Let M^n be a compact minimal submanifold of the unit sphere S^{n+p} with normal bundle v. We denote by $R^\perp$ the curvature tensor field corresponding to the normal connection $\nabla^\perp$ in the normal bundle v of M^n, and define $K^\perp : M \to R$ by

$$K^\perp = \sum_{i,j,\alpha,\beta} [R^\perp(e_i, e_j, N_\alpha, N_\beta)]^2,$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M^n and $\{N_1, \ldots, N_p\}$ is a local field of orthonormal normals. We call the function $K^\perp$ the normal curvature of the minimal submanifold M^n. In this paper we prove the following result.

THEOREM. *Let M^n be a compact minimal submanifold of S^{n+p}. If the normal curvature $K^\perp$, the scalar curvature S and the square of the length of the second fundamental form σ of M^n satisfy*

$$K^\perp \leq \sigma, \qquad S > (n-1)^2,$$

then M^n is totally geodesic.

This theorem can be considered as a partial generalization of the result of Shen [**6**, Theorem 3]. However, it will be an interesting question whether the condition $K^\perp \leq \sigma$ is redundant and Shen's result can be extended beyond dimension 3.

2. Preliminaries. Let M be a minimal submanifold of the unit sphere S^{n+p}, with normal bundle v. Then the second fundamental form h of M^n satisfies

$$(\nabla h)(X, Y, Z) = (\nabla h)(Y, Z, X) = (\nabla h)(Z, X, W), \qquad X, Y, Z \in \mathscr{X}(M), \tag{2.1}$$

where $\mathscr{X}(M)$ is the Lie algebra of smooth vector fields on M and $(\nabla h)(X, Y, Z)$ is defined by

$$(\nabla h)(X, Y, Z) = \nabla^\perp_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

where $\nabla^\perp$ is the connection defined in v and ∇ is the induced Riemannian connection with respect to the induced Riemannian metric g on M^n. The second covariant derivative $(\nabla^2 h)(X, Y, Z, W)$ of the second fundamental form is given by

$$(\nabla^2 h)(X, Y, Z, W) = \nabla_X^\perp(\nabla h)(Y, Z, W) - (\nabla h)(\nabla_X Y, Z, W) \\ - (\nabla h)(Y, \nabla_X Z, W) - (\nabla h)(Y, Z, \nabla_X W), \qquad X, Y, Z, W \in \mathscr{X}(M).$$

We have the following form of the Ricci identity

$$(\nabla^2 h)(X, Y, Z, W) - (\nabla^2 h)(Y, X, Z, W) = R^\perp(X, Y)h(Z, W) - h(R(X, Y)Z, W) \\ - h(Z, R(X, Y)W), \qquad X, Y, Z, W \in \mathscr{X}(M), \tag{2.2}$$

where $R^\perp$ and R are the curvature tensors of the connections $\nabla^\perp$ and ∇ respectively. Since M^n is a minimal submanifold for a local orthonormal frame $\{e_1, \ldots, e_n\}$ of M^n we have

$$\sum_{i=1}^n (\nabla h)(X, e_i, e_i) = 0, \\ \sum_{i=1}^n (\nabla^2 h)(X, Y, e_i, e_i) = 0. \tag{2.3}$$

Using the Ricci tensor Ric, we define the symmetric operator R^* by

$$\mathrm{Ric}(X, Y) = g(R^*(X), Y), X, Y \in \mathscr{X}(M).$$

Then the Gauss equation gives

$$A_{h(Y,Z)}X = R(X, Y)Z + A_{h(X,Z)}Y - g(Y, Z)X + g(X, Z)Y, \tag{2.4}$$

$$R^*(X) = (n-1)X - \sum_{i=1}^n A_{h(e_i,X)e_i}, \qquad X, Y, Z \in \mathscr{X}(M), \tag{2.5}$$

where A_N, $N \in \nu$, is the Weingarten map with respect to the normal N, satisfying $g(A_N X, Y) = g(h(X, Y), N)$. We define

$$\sigma = \sum_{i,j} \|h(e_i, e_j)\|^2, \\ \|A_h\|^2 = \sum_{i,j,k} \|A_{h(e_i,e_j)}e_k\|^2, \\ \|\nabla h\|^2 = \sum_{i,j,k} \|(\nabla h)(e_i, e_j, e_k)\|^2. \tag{2.6}$$

Now we prove the following lemma.

LEMMA. *Let M^n be a minimal submanifold of S^{n+p}, then for a local orthonormal frame $\{e_1, \ldots, e_n\}$, we have*

$$\sum_{i,j,k} R(e_k, e_i; e_j, A_{h(e_i,e_j)}e_k) = -\sigma + \|A_h\|^2 + \tfrac{1}{2}K^\perp - \sum_{i,j,\alpha,\beta} g(A_\alpha e_i, A_\beta e_j)^2,$$

where $A_\alpha \equiv A_{N_\alpha}$ and $\{N_1, \ldots, N_p\}$ is a local field of orthonormal normals.

Proof. Using the Ricci equation

$$R^\perp(X, Y; N_1, N_2) = g([A_{N_1}, A_{N_2}](X), Y), \qquad X, Y \in \mathscr{X}(M), N_1, N_2 \in \nu,$$

we get

$$K^{\perp} = \sum_{i,j,\alpha,\beta} [R^{\perp}(e_i, e_j; N_\alpha, N_\beta)]^2 = \sum_{i,j,\alpha,\beta} [g(A_\alpha A_\beta e_i, e_j) - g(A_\beta A_\alpha e_i, e_j)]^2$$

$$= 2 \sum_{i,j,\alpha,\beta} g(A_\alpha e_i, A_\beta e_j)^2 - 2 \sum_{i,j,\alpha,\beta} g(A_\alpha A_\beta e_i, e_j) g(A_\beta A_\alpha e_i, e_j), \tag{2.7}$$

since $\sum_{i,j,\alpha,\beta} g(A_\alpha e_j, A_\beta e_i)^2 = \sum_{i,j,\alpha,\beta} g(A_\beta e_j, A_\alpha e_i)^2$ which follows from the symmetry of A_α and A_β. Next using the Gauss equation, we have

$$\begin{aligned} R(e_k, e_i; e_j, A_{h(e_i,e_j)}e_k) &= \delta_{ij} g(h(e_k, e_k), h(e_i, e_j)) - \delta_{kj} g(h(e_i, e_j), h(e_i, e_k)) \\ &\quad + g(h(e_i, e_j), h(e_k, A_{h(e_i,e_j)}e_k)) - g(h(e_k, e_j), h(e_i, A_{h(e_i,e_j)}e_k)) \end{aligned} \tag{2.8}$$

since $A_{h(e_i,e_j)}e_k = \sum_\alpha g(A_\alpha e_i, e_j) A_\alpha e_k$, we obtain

$$\sum_{i,j,k} g(h(e_i, e_j), h(e_k, A_{h(e_i,e_j)}e_k)) = \sum_{i,j,k} g(A_{h(e_i,e_j)}e_k, A_{h(e_i,e_j)}e_k) = \|A_h\|^2 \tag{2.9}$$

and

$$\sum_{i,j,k} g(h(e_k, e_j), h(e_i, A_{h(e_i,e_j)}e_k)) = \sum_{i,j,\alpha,\beta} g(A_\alpha A_\beta e_i, e_j) g(A_\beta A_\alpha e_i, e_j). \tag{2.10}$$

Then using (2.7), (2.9) and (2.10) in (2.8) and using minimality of M^n we find

$$\sum_{i,j,k} R(e_k, e_i; e_j, A_{h(e_i,e_j)}e_k) = -\sigma + \|A_h\|^2 - \sum_{i,j,\alpha,\beta} g(A_\alpha e_i, A_\beta e_j)^2 + \tfrac{1}{2}K^{\perp}$$

which proves the lemma.

3. Proof of the theorem. Let M^n be a compact minimal submanifold of S^{n+p} satisfying the hypothesis of the theorem. Define $F: M \to R$ by $F = \frac{1}{2}\sigma$. Then it is straightforward to compute the Laplacian ΔF of the function F as

$$\Delta F = \sum_{i,j,k} g((\nabla^2 h)(e_k, e_k, e_i, e_j), h(e_i, e_j)) + \sum_{i,j,k} \|(\nabla h)(e_i, e_j, e_k)\|^2.$$

Using the Ricci identity (2.2) and equations (2.1) in above equation we arrive at

$$\begin{aligned} \Delta F = \sum_{i,j,k} [&R^{\perp}(e_k, e_i; h(e_k, e_j), h(e_i, e_j)) - R(e_k, e_i; e_k, A_{h(e_j,e_i)}e_j) \\ &- R(e_k, e_i; e_j, A_{h(e_i,e_j)}e_k)] + \|\nabla h\|^2. \end{aligned}$$

We employ (2.4) in the Ricci equation, to compute

$$\begin{aligned} R^{\perp}(e_k, e_i; h(e_k, e_j), h(e_i, e_j)) &= g(A_{h(e_i,e_j)}e_k, A_{h(e_i,e_j)}e_k \\ &\quad + R(e_i, e_k)e_j - \delta_{kj} e_i + \delta_{ij} e_k) - g(A_{h(e_k,e_j)}e_k, A_{h(e_i,e_j)}e_i) \end{aligned}$$

or

$$\begin{aligned} \sum_{i,j,k} R^{\perp}(e_k, e_i; h(e_k, e_j), h(e_i, e_j)) &= \|A_h\|^2 - \sigma + \sum_{i,j,k} R(e_i, e_k; e_j, A_{h(e_i,e_j)}e_k) \\ &\quad - g(A_{h(e_k,e_j)}e_k, A_{h(e_i,e_j)}e_i). \end{aligned} \tag{3.2}$$

Since (2.5) gives $R^*(e_j) = (n-1)e_j - \sum_k A_{h(e_k,e_j)}e_k$, we have

$$\begin{aligned} \sum_{i,j,k} g(A_{h(e_k,e_j)}e_k, A_{h(e_i,e_j)}e_i) &= \sum_{i,j} g((n-1)e_j - R^*(e_j), A_{h(e_i,e_j)}e_i) \\ &= (n-1)\sigma - \sum_{i,j} \mathrm{Ric}(e_j, A_{h(e_i,e_j)}e_i) \end{aligned}$$

$$= (n-1)\sigma - \sum_{i,j,k} R(e_k, e_j, A_{h(e_i,e_j)}e_i, e_k)$$
$$= (n-1)\sigma + \sum_{i,j,k} R(e_k, e_i, e_k, A_{h(e_i,e_j)}e_j). \tag{3.3}$$

Thus using (3.3) in (3.2), we have

$$\sum_{i,j,k} R^{\perp}(e_k, e_i; h(e_k, e_j), h(e_i, e_j)) = -n\sigma + \|A_h\|^2 + \sum_{i,j,k} [R(e_i, e_k; e_j, A_{h(e_i,e_j)}e_k) - R(e_k, e_i, e_k, A_{h(e_i,e_j)}e_j)]. \tag{3.4}$$

Using (3.4) in (3.1), we obtain

$$\Delta F = -n\sigma + \|A_h\|^2 - 2\sum_{i,j,k} [R(e_k, e_i; e_j, A_{h(e_i,e_j)}e_k) - R(e_k, e_i, e_k; A_{h(e_i,e_j)}e_j] + \|\nabla h\|^2. \tag{3.5}$$

Also, we have

$$\begin{aligned}
\sum_{i,j,k} R(e_k, e_i; e_k, A_{h(e_i,e_j)}e_j) &= -\sum_{i,j} \mathrm{Ric}(e_i, A_{h(e_i,e_j)}e_j) \\
&= -\sum_{i,j} g(R^*e_i, A_{h(e_i,e_j)}e_j) \\
&= -\sum_{i,j,\alpha} g(R^*e_i, A_\alpha e_j) g(A_\alpha e_i, e_j) \\
&= -\sum_{i,j,\alpha} g(R^*A_\alpha e_j, e_i) g(A_\alpha e_j, e_i) \\
&= -\sum_{j,\alpha} g(R^*A_\alpha e_j, A_\alpha e_j) \\
&= -\sum_{j,\alpha} \mathrm{Ric}(A_\alpha e_j, A_\alpha e_j) \\
&= -\sum_{j,\alpha} (n-1) g(A_\alpha e_j, A_\alpha e_j) + \sum_{i,j,\alpha} \|h(e_i, A_\alpha e_j)\|^2 \\
&= -(n-1)\sigma + \sum_{i,j,\alpha,\beta} g(A_\beta e_i, A_\alpha e_j)^2.
\end{aligned} \tag{3.6}$$

Using (3.6) and the lemma in Section 2 in (3.5), we obtain

$$\Delta F = (n-1)\sigma - \|A_h\|^2 + (\sigma - K^{\perp}) + \|\nabla h\|^2. \tag{3.7}$$

Now using the facts that

$$\begin{aligned}
\|A_h\|^2 &= \sum_{i,j,k} \|A_{h(e_i,e_j)}e_k\|^2 = \sum_{i,j,k,\alpha} g(A_\alpha e_i, e_j)^2 \|A_\alpha e_k\|^2 \\
&= \sum_{i,j,\alpha} g(A_\alpha e_i, e_j)^2 \|A_\alpha\|^2 = \sum_\alpha \|A_\alpha\|^2 \|A_\alpha\|^2 = \sum_\alpha \|A_\alpha\|^4
\end{aligned}$$

and $\sigma = \sum_\alpha \|A_\alpha\|^2$, in (3.7) and integrating it over M^n we obtain

$$\int_M \left\{ \sum_\alpha [(n-1) - \|A_\alpha\|^2] \|A_\alpha\|^2 + (\sigma - K^{\perp}) + \|\nabla h\|^2 \right\} dv = 0. \tag{3.8}$$

From the hypothesis of the theorem $S > (n-1)^2$, it follows that

$$n(n-1) - \sum_{\alpha} \|A_\alpha\|^2 > (n-1)^2,$$

that is, $\sum_{\alpha} \|A_\alpha\|^2 < (n-1)$, consequently $\|A_\alpha\|^2 < (n-1)$, and that $K^{\perp} \leq \sigma$. Thus in order for (3.8) to hold we must have $\|A_\alpha\| = 0$, that is M^n is totally geodesic.

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