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Mean-square stability of a class of stochastic integral equations

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The object of this paper is to investigate under very general conditions the existence and mean-square stability of a random solution of a class of stochastic integral equations in the form

$$x(t;\ \omega) = h\big(t,\ x(t;\ \omega)\big) + \int_0^t K(t,\ s;\ \omega) f\big(s,\ x(s;\ \omega)\big) ds \ ,$$

for $t \geq 0$, where a random solution is a second order stochastic process $\{x(t;\ \omega) : t \geq 0\}$ which satisfies the equation almost certainly. A random solution $x(t;\ \omega)$ is defined to be *stable in mean-square* if $E\left[|x(t;\ \omega)|^2\right] \leq \rho$ for all $t \geq 0$ and some $\rho > 0$ or *exponentially stable in mean-square* if $E\left[|x(t;\ \omega)|^2\right] \leq \rho e^{-\alpha t}$, $t \geq 0$, for some constants $\rho > 0$ and $\alpha > 0$.

1. Introduction

Two important types of stochastic integral equations currently under study by mathematicians and probabilists are those involving Îto or Îto-Doob type stochastic integrals and those involving mean-square integrals of second order stochastic processes. Stochastic integral equations involving the latter type of integrals occur in many probabilistic models for physical phenomena. For example, such equations arise in telephone traffic theory [5], [13], stochastic models for chemotherapy [11], [12], stochastic theory of turbulence [14], systems

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theory [9], [15], [16], [18], and in many other situations [1], [2].

The purpose of this paper is to study the concept of *stability in mean-square* of random solutions of a certain class of stochastic integral equations involving second order stochastic processes. Specifically, we shall consider stochastic integral equations of the Hammerstein type in the form

$$(1.1) \qquad x(t;\omega) = h\big(t, x(t;\omega)\big) + \int_0^t K(t, s;\omega) f\big(s, x(s;\omega)\big)\,ds ,$$

where $t \geq 0$ and

(i) (Ω, A, P) is a probability measure space, P is a probability measure defined on the sigma-algebra A of subsets of Ω, and $\omega \in \Omega$;

(ii) $x(t;\omega)$ is the unknown random variable for each $t \in R_+ = [0, \infty)$;

(iii) $h(t, x)$ is a function defined for each $t \in R_+$ and $x \in R$, the real numbers;

(iv) $K(t, s;\omega)$ is the stochastic kernel defined for $0 \leq s \leq t < \infty$ and $\omega \in \Omega$; and

(v) $f(t, x)$ is a function of $t \in R_+$ and $x \in R$.

Further conditions on the functions in equation (1.1) will be given in Section 2.

In [8], [17], and [19] equations similar to equation (1.1) were investigated. In this paper we shall study the existence, uniqueness, and mean-square stability properties of random solutions of (1.1) under more general conditions on the random functions h, K, and f than those given in [8], [17], and [19]. The main results will be given in Theorems 3.1 and 3.2, and certain special cases of the main theorems will be presented in Section 4. Also, it should be noted that the stochastic Volterra integral equations in [17] and [19] may be obtained as special cases of (1.1) whenever $h\big(t, x(t;\omega)\big)$ is replaced by a function $h(t;\omega)$ which does not depend on $x(t;\omega)$.

In order to obtain the objectives of this paper the concept of "admissibility" of Banach spaces [4] will be utilized.

2. Preliminaries

We shall make the following assumptions concerning the functions which appear in the stochastic integral equation (1.1):

(a) for each $t \in R_+$, $x(t; \omega) \in L_2(\Omega, A, P)$, that is, $\{x(t; \omega) : t \in R_+\}$ is a second order stochastic process on R_+, [7];

(b) for each $t \in R_+$, $h(t, x(t; \omega))$ and $f(t, x(t; \omega))$ are in the space $L_2(\Omega, A, P)$;

(c) the mapping $(t, s) \to K(t, s; \omega)$ is continuous from the set

$$\Delta = \{(t, s) : 0 \le s \le t < \infty\}$$

into the space $L_\infty(\Omega, A, P)$. Thus, for each $(t, s) \in \Delta$, we have

$$K(t, s; \omega)f(s, x(s; \omega)) \in L_2(\Omega, A, P) .$$

For each $(t, s) \in \Delta$ we shall denote the norm of $K(t, s; \omega)$ in $L_\infty(\Omega, A, P)$ by

$$|||K(t, s; \omega)||| = P - \operatorname*{ess\,sup}_{\omega \in \Omega} |K(t, s; \omega)|$$

and the norm of $x(t; \omega)$ in $L_2(\Omega, A, P)$ for each $t \in R_+$ by

$$\begin{aligned} \|x(t; \omega)\|_{L_2(\Omega,A,P)} &= \{E[|x(t; \omega)|^2]\}^{\frac{1}{2}} \\ &= \left\{\int_\Omega |x(t; \omega)|^2 dP(\omega)\right\}^{\frac{1}{2}} . \end{aligned}$$

DEFINITION 2.1. We define the space $C_c = C_c(R_+, L_2(\Omega, A, P))$ to be the space of all continuous functions from R_+ into the space $L_2(\Omega, A, P)$ with the topology of uniform convergence on every interval $[0, Q]$, $Q > 0$.

Note that C_c is a locally convex space with the topology defined by the following sequence of seminorms:

$$\|x(t;\ \omega)\|_n = \sup_{0\le t\le n} \|x(t;\ \omega)\|_{L_2(\Omega,A,P)}\ ,\quad n = 1,\ 2,\ \dots\ .$$

DEFINITION 2.2. We shall denote by $C_g = C_g\big(R_+,\ L_2(\Omega,\ A,\ P)\big)$ the space of all continuous functions from R_+ into $L_2(\Omega,\ A,\ P)$ such that there is a positive constant Γ and a positive-valued continuous function g on R_+ satisfying

$$\|x(t;\ \omega)\|_{L_2(\Omega,A,P)} \le \Gamma g(t)\ ,\quad t \in R_+\ .$$

The norm in $C_g\big(R_+,\ L_2(\Omega,\ A,\ P)\big)$ is defined by

$$\|x(t;\ \omega)\|_{C_g} = \sup_{t\in R_+} \left\{\frac{\|x(t;\omega)\|_{L_2(\Omega,A,P)}}{g(t)}\right\}\ .$$

When the function $g(t) = 1$ for all $t \in R_+$ in Definition 2.2, we obtain the space of all bounded continuous functions from R_+ into $L_2(\Omega,\ A,\ P)$, which will be denoted by $C = C\big(R_+,\ L_2(\Omega,\ A,\ P)\big)$. The space C is the space of all second order stochastic processes defined on R_+ which are bounded in mean-square and continuous in mean-square.

Throughout this paper B and D will denote Banach spaces such that $B,\ D \subset C_c\big(R_+,\ L_2(\Omega,\ A,\ P)\big)$.

DEFINITION 2.3. A pair of Banach spaces $(B,\ D)$ is said to be *admissible* with respect to a linear operator T if $T(B) \subset D$.

A lemma in [17] states that if the spaces B and D have stronger topologies than the space $C_c\big(R_+,\ L_2(\Omega,\ A,\ P)\big)$ and the pair $(B,\ D)$ is admissible with respect to a continuous linear operator $T : C_c \to C_c$, then T is continuous from B into D. Thus, T is bounded and for each $x(t;\ \omega) \in B$, there exists a constant $M > 0$ such that

$$\|(Tx)(t;\ \omega)\|_B \le M\|x(t;\ \omega)\|_D\ .$$

By a *random solution* of equation (1.1) we will mean a second order stochastic process on R_+ which satisfies the equation P almost everywhere.

In order to study the existence of a random solution of equation (1.1) we will employ the following fixed-point theorem, which was proven in [*10*]. Recall that an operator T on a Banach space B into itself is a nonlinear contraction if for all $x, y \in B$,

$$\|Tx-Ty\|_B \le \phi(\|x-y\|_B) ,$$

where ϕ is a real-valued continuous function satisfying $\phi(s) < s$ for $s > 0$.

THEOREM 2.1 (Nashed and Wong [*10*]). *Let S be a closed, bounded convex subset of a Banach space and let U and V be operators on S satisfying:*

(i) $U(x) + V(y) \in S$ for all $x, y \in S$;

(ii) U is a completely continuous operator on S;

(iii) V is a nonlinear contraction on S.

Then there exists a point $x^* \in S$ such that $U(x^*) + V(x^*) = x^*$.

Also, a fixed-point theorem of Boyd and Wong [*3*] which is a special case of Theorem 2.1 will be utilized to obtain the existence of a unique random solution of equation (1.1).

THEOREM 2.2 (Boyd and Wong [*3*]). *If V is a nonlinear contraction on a Banach space B, then V has a unique fixed point in B.*

Finally, we shall make the following definitions concerning the asymptotic behavior of a random solution of (1.1).

DEFINITION 2.4. A random solution of equation (1.1) is said to be

(a) *stable in mean-square* if there exists a positive constant ρ such that $E[|x(t;\omega)|^2] \le \rho$ for all $t \in R_+$, or

(b) *exponentially stable in mean-square* if there exist positive constants ρ and α such that $E[|x(t;\omega)|^2] \le \rho e^{-\alpha t}$ for all $t \in R_+$.

3. Existence of a random solution

We shall now prove two general theorems concerning the existence of a random solution of equation (1.1).

THEOREM 3.1. *Consider the stochastic integral equation* (1.1) *subject to the following conditions:*

(i) B *and* D *are Banach spaces stronger than* $C_c(R_+, L_2(\Omega, A, P))$ *and* (B, D) *is admissible with respect to the integral operator*

$$(Tx)(t;\omega) = \int_0^t K(t, s;\omega)x(s;\omega)ds \ , \quad t \geq 0 \ ,$$

where K *behaves as described in Section 2;*

(ii) $x(t;\omega) \to f(t, x(t;\omega))$ *is an operator from*

$$S = \{x(t;\omega) \in D : \|x(t;\omega)\|_D \leq \rho\}$$

into B *satisfying*

$$\|f(t, x(t;\omega))-f(t, y(t;\omega))\|_B \leq \phi(\|x(t;\omega)-y(t;\omega)\|_D) \ ,$$

where $\rho > 0$ *is a constant and* ϕ *is a real-valued continuous function such that* $\phi(s) < s$ *for* $s > 0$ *;*

(iii) $x(t;\omega) \to h(t, x(t;\omega))$ *is a completely continuous operator from* S *into itself such that for some* $\gamma > 0$ *,* $\|h(t, x(t;\omega))\|_D \leq \gamma$ *.*

Then there exists in S *a random solution of equation* (1.1)*, provided* $M_T < 1$ *and*

$$\gamma + M_T\|f(t, 0)\|_B \leq \rho(1-M_T) \ ,$$

where M_T *is the norm of* T *.*

Proof. Define the operator V from S into D by

$$(Vx)(t;\omega) = \int_0^t K(t, s;\omega)f(s, x(s;\omega))ds \ . \tag{3.1}$$

Let $x(t;\omega)$, $y(t;\omega) \in S$. Then

$$\|h(t, x(t; \omega))+(Vy)(t; \omega)\|_D \le \|h(t, x(t; \omega))\|_D + M_T\|f(t, y(t; \omega))\|_B$$

by condition *(i)* of the theorem and the remark following Definition 2.3. But, since

$$\|f(t, y(t; \omega))\|_B \le \phi(\|y(t; \omega)-0\|_D) + \|f(t, 0)\|_B$$

by condition *(ii)*, we have from condition *(iv)* that

$$\begin{aligned}(3.2)\qquad \|h(t, x(t; \omega))+(Vy)(t; \omega)\|_D &\le \|h(t, x(t; \omega))\|_D \\ &\quad + M_T\phi(\|y(t; \omega)\|_D) + M_T\|f(t, 0)\|_B \\ &\le \gamma + M_T\|f(t, 0)\|_B + M_T\phi(\|y(t; \omega)\|_D) .\end{aligned}$$

Now, from condition *(ii)* and the last hypothesis of the theorem, we obtain from inequality (3.2),

$$\begin{aligned}\|h(t, x(t; \omega))+(Vy)(t; \omega)\|_D &< \rho(1-M_T) + M_T\|y(t; \omega)\|_D \\ &\le \rho(1-M_T) + M_T\rho \\ &= \rho .\end{aligned}$$

That is, $h(t, x(t; \omega)) + (Vy)(t; \omega) \in S$ and condition *(i)* of Theorem 2.1 is satisfied.

Furthermore,

$$\begin{aligned}\|(Vx)(t; \omega)-(Vy)(t; \omega)\|_D &= \left\|\int_0^t K(t, s; \omega)[f(s, x(s; \omega))-f(s, y(s; \omega))]\,ds\right\|_D \\ &\le M_T\|f(t, x(t; \omega))-f(t, y(t; \omega))\|_B \\ &\le \phi(\|x(t; \omega)-y(t; \omega)\|_D)\end{aligned}$$

by condition *(ii)* and the hypothesis that $M_T < 1$. Thus, V is a nonlinear contraction operator on S .

Therefore, by condition *(iii)* and Theorem 2.1 there exists in S a random solution of (1.1).

By imposing a slightly stronger condition on the stochastic process $h(t, x(t; \omega))$, we can obtain the existence of a *unique* random solution of (1.1).

THEOREM 3.2. *Suppose the stochastic integral equation* (1.1)

satisfies the following conditions:

(i) *same as condition (i) of Theorem* 3.1*;*

(ii) *same as condition (ii) of Theorem* 3.1*;*

(iii) $x(t;\omega) \to h(t, x(t;\omega))$ *is an operator from the set* S *into* D *satisfying*

$$\|h(t, x(t;\omega))-h(t, y(t;\omega))\|_D \le \gamma\phi(\|x(t;\omega)-y(t;\omega)\|_D) ,$$

where $\gamma > 0$ *is a constant.*

Then there exists in S *a unique random solution of equation* (1.1)*, provided that* $\gamma + M_T < 1$ *and*

$$\|h(t, 0)\|_D + M_T\|f(t, 0)\|_B \le \rho(1-\gamma-M_T) ,$$

where M_T *is the norm of the operator* T *.*

Proof. Define the operator V from S into D by

$$(Vx)(t;\omega) = h(t, x(t;\omega)) + \int_0^t K(t, s;\omega)f(s, x(s;\omega))ds .$$

Let $x(t;\omega) \in S$. We must show that V is a nonlinear contraction on S into itself. By conditions *(i)* and *(ii)* of the theorem

$$(3.3) \quad \|(Vx)(t;\omega)\|_D \le \|h(t, x(t;\omega))\|_D + \left\|\int_0^t K(t, s;\omega)f(s, x(s;\omega))ds\right\|_D$$
$$\le \|h(t, x(t;\omega))\|_D + M_T\|f(t, x(t;\omega))\|_B .$$

But since

$$\|h(t, x(t;\omega))\|_D \le \|h(t, x(t;\omega))-h(t, 0)\|_D + \|h(t, 0)\|_D$$

and a similar inequality holds for $f(t, x(t;\omega))$, we have from inequality (3.3) and conditions *(ii)* and *(iii)* that

$$\|(Vx)(t;\omega)\|_D \le \gamma\phi(\|x(t;\omega)\|_D) + \|h(t, 0)\|_D + M_T\phi(\|x(t;\omega)\|_D) + M_T\|f(t, 0)\|_B$$
$$< (\gamma+M_T)\|x(t;\omega)\|_D + \|h(t, 0)\|_D + M_T\|f(t, 0)\|_B$$
$$\le \rho(\gamma+M_T) + \rho(1-\gamma-M_T) = \rho$$

by the last hypothesis of the theorem. Hence, $V(S) \subset S$.

Now, let $x(t;\omega), y(t;\omega) \in S$. Then applying the hypotheses of the theorem, we have

$$\begin{aligned}
&\|(Vx)(t;\omega)-(Vy)(t;\omega)\|_D \\
&\quad \le \|h(t, x(t;\omega))-h(t, y(t;\omega))\|_D \\
&\qquad\qquad + \left\|\int_0^t K(t, s;\omega)[f(s, x(s;\omega))-f(s, y(s;\omega))]\,ds\right\|_D \\
&\quad \le \|h(t, x(t;\omega))-h(t, y(t;\omega))\|_D + M_T\|f(t, x(t;\omega))-f(t, y(t;\omega))\|_B \\
&\quad \le \gamma\phi(\|x(t;\omega)-y(t;\omega)\|_D) + M_T\phi(\|x(t;\omega)-y(t;\omega)\|_D) \\
&\quad \le \phi(\|x(t;\omega)-y(t;\omega)\|_D)
\end{aligned}$$

since $\gamma + M_T < 1$. Thus, V is a nonlinear contraction on S into itself, and by Theorem 2.2 there exists a unique random solution of equation (1.1), $x^*(t;\omega) \in S$.

4. Mean-square stability

By choosing the Banach spaces B and D to be the spaces $C(R_+, L_2(\Omega, A, P))$ or $C_g(R_+, L_2(\Omega, A, P))$, we may obtain certain useful special cases of Theorems 3.1 and 3.2 concerning the existence and mean-square stability of random solutions of (1.1).

Recall that an operator h on the bounded set S is completely continuous if the image $h(S)$ is relatively compact, that is, if every bounded sequence in $h(S)$ has a convergent subsequence.

Let $x(t;\omega) \to h(t, x(t;\omega))$ be an operator from the set

$$S = \{x(t;\omega) \in C : \|x(t;\omega)\|_C \le \rho\} \tag{4.1}$$

into itself, where $\rho > 0$ is a constant. From [6] it follows that $h(S) \subset C$ is relatively compact if h is such that

$$\text{(A)} \quad \lim_{\delta\to 0^+} \sup_{x\in S}\left\{\sup_{\substack{s,t\ge 0\\ |s-t|<\delta}} [\|h(t, x(t;\omega))-h(s, x(s;\omega))\|_{L_2(\Omega,A,P)}]\right\} = 0$$

and

(B) for each fixed $t \in R_+$ every sequence in $\{h(t, x(t; \omega)) : x(t; \omega) \in S\} \subset L_2(\Omega, A, P)$ has a convergent subsequence.

We shall now prove the following special case of Theorem 3.1.

THEOREM 4.1. *Consider the stochastic integral equation* (1.1) *under the following conditions:*

(i) *there exist a positive number* Γ *and a positive continuous function* $g(t)$ *on* R_+ *such that*

$$\int_0^t |||K(t, s; \omega)||| \, g(s)ds \le \Gamma \ , \quad t \in R_+ \ ;$$

(ii) $x(t; \omega) \to f(t, x(t; \omega))$ *is an operator from the set* S *given by* (4.1) *into* $C_g(R_+, L_2(\Omega, A, P))$ *satisfying for every* $t \in R_+$,

$$|f(t, x(t; \omega))-f(t, y(t; \omega))| \le g(t)\phi(\|x(t; \omega)-y(t; \omega)\|_C)$$

with probability one for $x(t; \omega), y(t; \omega) \in S$, *where* ϕ *is a real-valued continuous function such that* $\phi(s) < s$ *for* $s > 0$;

(iii) $x(t; \omega) \to h(t, x(t; \omega))$ *is an operator from* S *into* C *such that* $|h(t, x(t; \omega))| \le \gamma$ *with probability one for every* $t \in R_+$, *some* $\gamma > 0$ *and such that* (A) *and* (B) *above are satisfied.*

Then there exists a random solution $x^*(t; \omega)$ *of* (1.1) *which is stable in mean-square, provided* $\sup_{t \in R_+} |f(t, 0)|/g(t)$, Γ *and* γ *are sufficiently small.*

Proof. By condition *(i)* the pair of Banach spaces (C_g, C) is admissible with respect to the integral operator

$$(4.2) \qquad (Tx)(t;\ \omega) = \int_0^t K(t,\ s;\ \omega)x(s;\ \omega)ds\ ,$$

since for $x(t;\ \omega) \in C_g$,

$$\begin{aligned}\|(Tx)(t;\ \omega)\|_{L_2(\Omega,A,P)} &\le \int_0^t |||K(t,\ s;\ \omega)||| \cdot \|x(s;\ \omega)\|_{L_2(\Omega,A,P)}ds \\ &\le \sup_{t\in R_+} \left\{\frac{\|x(t;\omega)\|_{L_2(\Omega,A,P)}}{g(t)}\right\} \int_0^t |||K(t,\ s;\ \omega)|||\, g(s)ds \\ &\le \|x(t;\ \omega)\|_{C_g} \cdot \Gamma\end{aligned}$$

by definition of the norm in $C_g\big(R_+,\ L_2(\Omega,\ A,\ P)\big)$. That is, $(Tx)(t;\ \omega) \in C$.

It clearly follows from condition *(ii)* that for $x(t;\ \omega),\ y(t;\ \omega) \in S$,

$$\|f\big(t,\ x(t;\ \omega)\big)-f\big(t,\ y(t;\ \omega)\big)\|_{C_g} \le \phi\big(\|x(t;\ \omega)-y(t;\ \omega)\|_C\big)\ ,$$

and from condition *(iii)* that h is completely continuous from S into itself provided γ is small enough.

Hence, by Theorem 3.1 there exists a random solution $x^*(t;\ \omega) \in C$ of equation (1.1) satisfying

$$E\big[|x^*(t;\ \omega)|^2\big] \le \rho^2\ ,\quad t \in R_+\ ,$$

whenever Γ, γ and $\sup_{t\in R_+} |f(t,\ 0)|/g(t)$ are small enough.

Note that by Jensen's inequality $x^*(t;\ \omega)$ is also *stable in the mean*

$$E[\,|x^*(t;\ \omega)|\,] \le \rho\ ,\quad t \in R_+\ .$$

COROLLARY 4.2. *Consider the stochastic integral equation* (1.1) *subject to the following conditions:*

(i) $\int_0^t |||K(t,\ s;\ \omega)|||\, ds \le \Lambda$, $t \in R_+$, *where* Λ *is some positive constant;*

(ii) $x(t,\ \omega) \to f\big(t,\ x(t;\ \omega)\big)$ *is an operator from the set* S *given*

by (4.1) *into* $C(R_+, L_2(\Omega, A, P))$ *such that for each* $t \in R_+$,

$$|f(t, x(t;\omega))-f(t, y(t;\omega))| \le \phi(\|x(t;\omega)-y(t;\omega)\|_C)$$

with probability one for $x(t;\omega), y(t;\omega) \in S$, *where* ϕ *is as given in Theorem* 4.1 *(ii);*

(iii) same as condition (iii) of Theorem 4.1.

Then there exists a random solution $x^*(t;\omega) \in C$ *of* (1.1) *which is stable in mean-square, provided* $\|f(t, 0)\|_C$, Λ *and* γ *are small enough.*

Proof. Take $g(t) = 1$ for all $t \in R_+$ in Theorem 4.1.

The following special case of Theorem 3.2 guarantees the existence of a unique random solution of (1.1) which is exponentially stable in mean-square.

THEOREM 4.3. *Assume that the stochastic integral equation* (1.1) *satisfies:*

(i) $|||K(t, s;\omega)||| \le \Gamma\exp[-\alpha(t-s)]$, $0 \le s \le t < \infty$, *where* $\Gamma > 0$ *and* $\alpha > 0$ *are constants;*

(ii) $x(t;\omega) \to f(t, x(t;\omega))$ *is an operator on*

$$S_g = \left\{x(t;\omega) \in C_g : \|x(t;\omega)\|_{C_g} \le \rho\right\}$$

with values in $C_g(R_+, L_2(\Omega, A, P))$ *satisfying for each* $t \in R_+$,

$$|f(t, x(t;\omega))-f(t, y(t;\omega))| \le e^{-\beta t}\phi\left(\|x(t;\omega)-y(t;\omega)\|_{C_g}\right) \tag{4.3}$$

with probability one for $x(t;\omega), y(t;\omega) \in S_g$, *where* $0 < \beta < \alpha$, $\rho > 0$ *is a constant,* ϕ *is as given in Theorem* (4.1) *(ii), and* $g(t) = e^{-\beta t}$, $t \ge 0$;

(iii) $x(t;\omega) \to h(t, x(t;\omega))$ *is an operator on* S_g *with values in* $C_g(R_+, L_2(\Omega, A, P))$ *satisfying for each* $t \in R_+$,

$$|h(t, x(t; \omega))-h(t, y(t; \omega))| \le e^{-\beta t}\gamma\phi\left(\|x(t; \omega)-y(t; \omega)\|_{C_g}\right)$$

with probability one for $x(t; \omega), y(t; \omega) \in S_g$ *, where* $\gamma > 0$ *is a constant.*

Then there exists a unique random solution of equation (1.1) *which is exponentially stable in mean-square, provided that* γ, Γ *and* $\|f(t, 0)\|_{C_g}$ *are sufficiently small.*

Proof. It is clear from condition *(ii)* that if inequality (4.3) is divided by $e^{-\beta t}$, we obtain for $x(t; \omega), y(t; \omega) \in S_g$,

$$\sup_{t\in R_+}\left\{\frac{\|f(t,x(t;\omega))-f(t,y(t;\omega))\|_{L_2(\Omega,A,P)}}{e^{-\beta t}}\right\} \le \phi\left(\|x(t; \omega)-y(t; \omega)\|_{C_g}\right)$$

or

$$\|f(t, x(t; \omega))-f(t, y(t; \omega))\|_{C_g} \le \phi\left(\|x(t; \omega)-y(t; \omega)\|_{C_g}\right)$$

when $g(t) = e^{-\beta t}$, $t \in R_+$. Similarly, from condition *(iii)* with $g(t) = e^{-\beta t}$, we have

$$\|h(t, x(t; \omega))-h(t, y(t; \omega))\|_{C_g} \le \gamma\phi\left(\|x(t; \omega)-y(t; \omega)\|_{C_g}\right) .$$

We need only to show that condition *(i)* implies that (C_g, C_g) is admissible with respect to the linear operator T given by equation (4.2). Let $x(t; \omega) \in C_g(R_+, L_2(\Omega, A, P))$ with $g(t) = e^{-\beta t}$, $t \ge 0$. Then from condition *(i)*,

$$\begin{aligned}
\|(Tx)(t;\ \omega)\|_{L_2(\Omega,A,P)} &\le \int_0^t |||K(t,\ s;\ \omega)||| \cdot \|x(s;\ \omega)\|_{L_2(\Omega,A,P)} ds \\
&\le \Gamma e^{-\alpha t} \int_0^t e^{\alpha s} \frac{\|x(s;\omega)\|_{L_2(\Omega,A,P)}}{e^{-\beta s}} e^{-\beta s} ds \\
&\le \Gamma e^{-\alpha t} \sup_{t \in R_+} \left\{ \frac{\|x(t;\omega)\|_{L_2(\Omega,A,P)}}{e^{-\beta t}} \right\} \int_0^t e^{(\alpha-\beta)s} ds \\
&\le \Gamma(\alpha-\beta)^{-1} \|x(t;\ \omega)\|_{C_g} e^{-\beta t} ,
\end{aligned}$$

which means that $(Tx)(t;\ \omega) \in C_g(R_+,\ L_2(\Omega,\ A,\ P))$.

Therefore, if Γ, γ and $\|f(t,\ 0)\|_{C_g}$ are sufficiently small, by Theorem 3.2 there exists a unique random solution of (1.1), $x^*(t;\ \omega) \in S_g$. That is,

$$E\left[|x^*(t;\ \omega)|^2\right] \le \rho^2 e^{-2\beta t} \ , \quad t \in R_+ ,$$

which means that $x^*(t;\ \omega)$ is exponentially stable in mean-square.

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