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ON BANACH SPACES OF VECTOR VALUED CONTINUOUS FUNCTIONS

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Let K be a compact Hausdorff space and let E be a Banach space. We denote by $C(K, E)$ the Banach space of all E-valued continuous functions defined on K, endowed with the supremum norm.

Recently, Talagrand [*Israel J. Math.* 44 (1983), 317-321] constructed a Banach space E having the Dunford-Pettis property such that $C([0, 1], E)$ fails to have the Dunford-Pettis property. So he answered negatively a question which was posed some years ago.

We prove in this paper that for a large class of compacts K (the scattered compacts), $C(K, E)$ has either the Dunford-Pettis property, or the reciprocal Dunford-Pettis property, or the Dieudonné property, or property V if and only if E has the same property.

Also some properties of the operators defined on $C(K, E)$ are studied.

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1. Introduction

In 1953 Grothendieck [4] axiomatized some relevant properties of $C(K)$ spaces, introducing among others the so-called Dunford-Pettis, reciprocal Dunford-Pettis and Dieudonné properties. Later Pelczynski [7], in 1962, showed that $C(K)$ spaces enjoy another property, that he called property V, which can be defined on analogous terms to the preceding ones. Since then, these properties began to be studied on spaces of vector valued continuous functions, in particular on $C(K, E)$. The problem that was posed is the following: if E has the Dunford-Pettis property, does $C(K, E)$ have this property too? The same question was asked for the other properties. This problem remained open for some years but, recently, Talagrand [9] constructed a Banach space E having the Dunford-Pettis property such that $C([0, 1], E)$ fails to have the Dunford-Pettis property. Talagrand's work shows the interest of looking for conditions on spaces K and E to obtain an affirmative answer to the posed problem.

We prove in this paper that for a large class of compacts K (the scattered compacts), it is enough to take a Banach space E having one of the mentioned properties to insure that $C(K, E)$ has the same property.

Since the study of such properties on a space is closely related to the study of the operators defined on it, we devote the first part of our work to show some properties of the operators defined on $C(K, E)$.

Throughout the paper E and F are Banach spaces, K is a compact Hausdorff space and Σ is the σ-field of Borel subsets of K. $C(K, E)$ is the Banach space of all E-valued continuous functions on K, and $B(\Sigma, E)$ is the Banach space of all functions $\varphi : K \to E$ which are the uniform limit of a sequence of Σ-simple functions. Both spaces are endowed with the supremum norm. The term "operator" means a bounded linear operator. We denote by $L(E, F)$ the space of all operators from E to F.

It is well known that an operator $T : C(K, E) \to F$ may be represented as an integral with respect to a finitely additive set function $m : \Sigma \to L(E, F'')$ having finite semivariation on K $\left(\hat{m}(K) < +\infty\right)$ and so that $\|T\| = \hat{m}(K)$ (see, for example, [2], p. 182); m is called the representing measure of T.

A compact space K is scattered if every subset A of K has a point relatively isolated in A . The class of compact scattered spaces includes all countable compact spaces and all compact ordinals (with the interval topology).

2. Some properties of the operators defined on $C(K, E)$

An operator $T : C(K, E) \to F$ whose representing measure m has its values in $L(E, F)$ determines an extension $\hat{T} : B(\Sigma, E) \to F$ given by

$$\hat{T}(\varphi) = \int_K \varphi dm \ , \quad \varphi \in B(\Sigma, E) \ ,$$

with $\|\hat{T}\| = \|T\|$ (see [1], Theorem 2).

Batt and Berg [1] showed that an operator $T : C(K, E) \to F$ is weakly compact if and only if its extension $\hat{T}$ to $B(\Sigma, E)$ is weakly compact.

We shall prove that one can obtain analogous results for other properties of T when K is metrizable.

THEOREM 1. *Let K be metrizable. Then an operator $T : C(K, E) \to F$ is unconditionally converging if and only if its extension $\hat{T}$ to $B(\Sigma, E)$ is unconditionally converging.*

Proof. Let $T : C(K, E) \to F$ be an unconditionally converging operator. Then, by Theorem 3 and Lemma 2 of [3], its representing measure m has its values in $L(E, F)$ and there is a finite non negative measure λ on Σ so that

$$\lim_{\lambda(A)\to 0} \hat{m}(A) = 0 \ . \tag{1}$$

If we suppose that $\hat{T} : B(\Sigma, E) \to F$ is not unconditionally converging, then there exist $\varepsilon > 0$ and a weakly unconditionally convergent series $\sum \varphi_n$ in $B(\Sigma, E)$ such that

$$\|\hat{T}(\varphi_n)\| > \varepsilon \quad \text{for all} \quad n \in \mathbb{N} \ . \tag{2}$$

It is well known that a series $\sum x_n$ in a Banach space is weakly unconditionally convergent if and only if the set

$$\left\{\sum_{n\in\sigma} x_n \ : \sigma \subset \mathbb{N} \text{ is finite}\right\}$$

is bounded. Therefore there is $M > 0$ verifying

(3) $$\left\|\sum_{n\in\sigma} \varphi_n\right\| < M \text{ for all finite subsets } \sigma \text{ of } \mathbb{N} .$$

By (1) we can choose $\delta > 0$, $\delta < \lambda(K)$, such that

(4) $$\hat{m}(A) < \frac{\varepsilon}{4M} \text{ for } A \in \Sigma \text{ with } \lambda(A) < \delta .$$

According to Lusin's theorem, for each $n \in \mathbb{N}$, there exists a compact $K_n \subset K$ such that $\lambda(K\backslash K_n) < \delta/2^n$ and $\varphi_n|_{K_n}$ (the restriction of φ_n to K_n) is continuous. Put $K_0 = \bigcap_{n=1}^{\infty} K_n$. Then $\lambda(K\backslash K_0) < \delta$, and $K_0 \neq \emptyset$ because $\delta < \lambda(K)$. Let us denote $\Phi_n = \varphi_n|_{K_0}$ for $n \in \mathbb{N}$. By (3) the series $\sum \Phi_n$ is weakly unconditionally convergent in $C(K_0, E)$.

Now, by the Borsuk-Dungundji theorem (see 21.1.4 of [8]), there is an operator $S : C(K_0, E) \to C(K, E)$, with $\|S\| = 1$, so that $S(\Phi)|_{K_0} = \Phi$ for every $\Phi \in C(K_0, E)$. The operator $TS : C(K_0, E) \to F$ is unconditionally converging. However, the series $\sum TS(\Phi_n)$ does not converge in F because, by (2), (3) and (4), for each $n \in \mathbb{N}$,

$$\begin{aligned}\|TS(\Phi_n)\| &= \left\|\int_K S(\Phi_n)\,dm\right\| \geq \left\|\int_{K_0} \varphi_n\,dm\right\| - \left\|\int_{K\backslash K_0} S(\Phi_n)\,dm\right\| \\ &\geq \left\|\int_K \varphi_n\,dm\right\| - \left\|\int_{K\backslash K_0} \varphi_n\,dm\right\| - \|S(\Phi_n)\|\hat{m}(K\backslash K_0) \\ &\geq \|\hat{T}(\varphi_n)\| - 2M\hat{m}(K\backslash K_0) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2} .\end{aligned}$$

This contradiction shows that if $T : C(K, E) \to F$ is unconditionally converging then $\hat{T}$ is also. The converse is obvious.

THEOREM 2. *Let K be metrizable. Then an operator $T : C(K, E) \to F$ transforms weakly Cauchy sequences into weakly convergent ones if and only if its extension $\hat{T}$ to $B(\Sigma, E)$ transforms weakly Cauchy sequences into weakly convergent ones.*

Proof. Let $T : C(K, E) \to F$ be an operator which maps weakly Cauchy sequences into weakly convergent sequences. Then T is unconditionally converging. Let m and λ be as in the preceding proof. Let (φ_n) be a weakly Cauchy sequence in $B(\Sigma, E)$ and let $y'' \in F''$ so that $(\hat{T}(\varphi_n))$ is $\sigma(F'', F')$-convergent to y''. If we suppose that $(\hat{T}(\varphi_n))$ is not weakly convergent in F then $y'' \notin F$. By using Grothendieck's completeness theorem (see 3.11.4 of [5]) it follows that there exist $\varepsilon > 0$ and a net $(y_i')_{i \in I}$ in the unit ball of F' which is $\sigma(F', F)$-convergent to zero such that

$$|\langle y_i', y'' \rangle| > \varepsilon \text{ for all } i \in I . \tag{5}$$

Choose $\delta > 0$, $\delta < \lambda(K)$, verifying

$$\hat{m}(A) < \frac{\varepsilon}{8 \sup \|\varphi_n\|} \text{ for } A \in \Sigma \text{ with } \lambda(A) < \delta .$$

Similarly as in the preceding proof we can take a non empty compact $K_0 \subset K$ so that $\lambda(K \setminus K_0) < \delta$ and $\Phi_n = \varphi_n|_{K_0}$ is continuous for $n \in \mathbb{N}$, and an operator $S : C(K_0, E) \to C(K, E)$, with $\|S\| = 1$, such that $S(\Phi)|_{K_0} = \Phi$ for $\Phi \in C(K_0, E)$. For each $t \in K_0$ the sequence $(\Phi_n(t))$ is weakly Cauchy in E, therefore, according to Theorem 9 of [3], (Φ_n) is weakly Cauchy in $C(K_0, E)$. Then $(TS(\Phi_n))$ is weakly convergent to an element $y \in F$. Since $(y_i')_{i \in I}$ is $\sigma(F', F)$-convergent to zero there exists $i_0 \in I$ so that

$$|\langle y, y_i' \rangle| < \varepsilon/6 \text{ for all } i \geq i_0 .$$

Let $i \geq i_0$; then there is $n \in \mathbb{N}$ verifying

$$|\langle \hat{T}(\varphi_n) - y'', y_i' \rangle| < \varepsilon/6 \text{ and } |\langle TS(\Phi_n) - y, y_i' \rangle| < \varepsilon/6 .$$

Thus we have

$$\begin{aligned}|\langle y'', y_i'\rangle| &\le |\langle y''-\hat{T}(\varphi_n), y_i'\rangle| + |\langle \hat{T}(\varphi_n)-TS(\Phi_n), y_i'\rangle| \\ &\qquad + |\langle TS(\Phi_n)-y, y_i'\rangle| + |\langle y, y_i'\rangle| \\ &< \frac{\varepsilon}{2} + \|y_i'\|\|\hat{T}(\varphi_n)-TS(\Phi_n)\| \\ &\le \frac{\varepsilon}{2} + \left\|\int_{K\setminus K_0} (\varphi_n - S(\Phi_n))\,dm\right\| \\ &\le \frac{\varepsilon}{2} + 2\|\varphi_n\|\hat{m}(K\setminus K_0) < \frac{3}{4}\varepsilon .\end{aligned}$$

But this contradicts (5).

The converse is clear.

THEOREM 3. *Let K be metrizable. Then an operator $T : C(K, E) \to F$ maps weakly convergent sequences into norm convergent sequences if and only if its extension $\hat{T}$ to $B(\Sigma, E)$ maps weakly convergent sequences into norm convergent ones.*

Proof. Let $T : C(K, E) \to F$ be an operator which maps weakly convergent sequences into norm convergent ones. Then T is unconditionally converging. Let m and λ be as in the proof of Theorem 1. Let (φ_n) be a sequence in $B(\Sigma, E)$ which is weakly convergent to zero. Suppose that there exist $\varepsilon > 0$ and a subsequence of (φ_n) (which we still denote by (φ_n)) so that

$$\|\hat{T}(\varphi_n)\| > \varepsilon \quad \text{for all} \quad n \in \mathbb{N} . \tag{6}$$

Choose $\delta > 0$, $\delta < \lambda(K)$, verifying

$$\hat{m}(A) < \frac{\varepsilon}{6\sup\|\varphi_n\|} \quad \text{for} \quad A \in \Sigma \quad \text{with} \quad \lambda(A) < \delta .$$

Reasoning as in the proof of Theorem 1, there exist a non empty compact $K_0 \subset K$, with $\lambda(K\setminus K_0) < \delta$, such that $\Phi_n = \varphi_n|_{K_0}$ is continuous for all $n \in \mathbb{N}$, and an operator $S : C(K_0, E) \to C(K, E)$, with $\|S\| = 1$, so that $S(\Phi)|_{K_0} = \Phi$ for $\Phi \in C(K_0, E)$. According to Theorem 9 of [3], (Φ_n) is weakly convergent to zero in $C(K_0, E)$. Then $(TS(\Phi_n))$ is norm convergent to zero and there exists $n_0 \in \mathbb{N}$ such that

$$\|TS(\Phi_n)\| < \varepsilon/3 \quad \text{for all} \quad n \geq n_0 .$$

Thus if $n \geq n_0$ one has

$$\begin{aligned}\|\hat{T}(\varphi_n)\| &\leq \|\hat{T}(\varphi_n)-TS(\Phi_n)\| + \|TS(\Phi_n)\| \\ &< \left\|\int_{K\setminus K_0} (\varphi_n-S(\Phi_n))\,dm\right\| + \frac{\varepsilon}{3} \\ &\leq 2\|\varphi_n\|\hat{m}(K\setminus K_0) + \frac{\varepsilon}{3} < \frac{2}{3}\varepsilon .\end{aligned}$$

But this contradicts (6).

The converse is obvious.

3. Some properties on $C(K, E)$

THEOREM 4. *If K is scattered then $C(K, E)$ has the Dunford-Pettis property if and only if E has.*

Proof. It is clear that if $C(K, E)$ has the Dunford-Pettis property then E has it too.

Suppose that E has the Dunford-Pettis property.

(A) If K is metrizable then, by 8.5.5 of [8], K is countable. Now the proof of Theorem 13 (a) of [3] works the same here.

(B) For a general K, let $T : C(K, E) \to F$ be a weakly compact operator and let (Φ_n) be a sequence in $C(K, E)$ weakly convergent to zero. Similarly as in [1], page 236, we can construct a metrizable quotient space $\overline{K}$ of K and a sequence $(\overline{\Phi}_n) \subset C(\overline{K}, E)$ such that $\overline{\Phi}_n(\pi(t)) = \Phi_n(t)$ for all $t \in K$ and $n \in \mathbb{N}$, where $\pi : K \to \overline{K}$ is the canonical mapping. By 8.5.3 of [8], $\overline{K}$ is scattered, and Theorem 9 of [3] implies that $(\overline{\Phi}_n)$ is weakly convergent to zero in $C(\overline{K}, E)$. If we consider the operator $\overline{T} : C(\overline{K}, E) \to F$ defined by $\overline{T}(\overline{\Phi}) = T(\overline{\Phi}\circ\pi)$ for $\overline{\Phi} \in C(\overline{K}, E)$, it follows from (A) that $\lim_n \|\overline{T}(\overline{\Phi}_n)\| = 0$. Since $T(\Phi_n) = \overline{T}(\overline{\Phi}_n)$ for all $n \in \mathbb{N}$, we conclude that $C(K, E)$ has the Dunford-Pettis property.

Note that if $C(K, E)$ has the Dunford-Pettis property when K is metrizable then, as in (B) of the preceding proof, it follows that $C(K, E)$ has the Dunford-Pettis property for every compact K. Therefore an immediate consequence of 8.5.7, 21.5.10 and 21.5.1 of [8], and Theorem 4 is the following:

COROLLARY 5. *$C(K, E)$ has the Dunford-Pettis property for every compact K if and only if $C([0, 1], E)$ has the Dunford-Pettis property.*

Recall that if m is the representing measure of an operator $T : C(K, E) \to F$, it is said that the semivariation $\hat{m}$ of m is continuous on Σ if for every decreasing sequence (A_n) in Σ, with $\bigcap_n A_n = \emptyset$, there is $\lim_n \hat{m}(A_n) = 0$.

LEMMA 6. *Let K be a metrizable scattered compact space and let $T : C(K, E) \to F$ be an operator whose representing measure m verifies*

(i) $m(\Sigma) \subset L(E, F)$,

(ii) $m(A) : E \to F$ *is weakly compact for each* $A \in \Sigma$,

(iii) $\hat{m}$ *is continuous on* Σ.

Then T is weakly compact.

Proof. By 8.5.5 of [8], K is countable. Put $K = \{t_i : i \in \mathbb{N}\}$. Let (Φ_n) be a bounded sequence in $C(K, E)$. For each $n \in \mathbb{N}$ we can take a Σ-simple function $\varphi_n \in B(\Sigma, E)$ so that $\|\varphi_n - \Phi_n\| < 1/n$. According to condition *(ii)*, for every $i \in \mathbb{N}$ the sequence $\big(m(\{t_i\})(\varphi_n(t_i))\big)_n$ has a weakly convergent subsequence. This fact enables us to use Cantor's diagonal argument to extract a subsequence of (φ_n) (which we still denote by (φ_n)) such that $\big(m(\{t_i\})(\varphi_n(t_i))\big)_n$ is weakly convergent in F for all $i \in \mathbb{N}$. For each $i \in \mathbb{N}$ let y_i be the $\sigma(F, F')$-limit of $\big(m(\{t_i\})(\varphi_n(t_i))\big)_n$. The series $\sum y_i$ converges in F. To prove this suppose that there exist $\varepsilon > 0$ and a sequence (σ_j) of finite subsets of $\mathbb{N}$, with $\max \sigma_j < \min \sigma_{j+1}$ for $j \in \mathbb{N}$, such that

$$\Big\|\sum_{i \in \sigma_j} y_i\Big\| > \varepsilon \quad \text{for all } j \in \mathbb{N}.$$

Hence for every $j \in \mathbb{N}$ we can choose y'_j in the unit ball of F' verifying

$$\left|\left\langle \sum_{i \in \sigma_j} y_i, y'_j \right\rangle\right| > \varepsilon .$$

Thus it follows from the choice of (y_i) that there is an increasing sequence $(n_j) \subset \mathbb{N}$ such that

$$\left|\left\langle \sum_{i \in \sigma_j} m(\{t_i\})(\varphi_{n_j}(t_i)), y'_j \right\rangle\right| > \varepsilon \quad \text{for } j \in \mathbb{N} .$$

We set $A_j = \bigcup_{k=j}^{\infty} \{t_i : i \in \sigma_k\}$, $j \in \mathbb{N}$. Then one has

$$\begin{aligned}\hat{m}(A_j) &\geq \hat{m}(\{t_i : i \in \sigma_j\}) \\ &\geq \frac{1}{\sup\|\varphi_n\|} \left\| \sum_{i \in \sigma_j} m(\{t_i\})(\varphi_{n_j}(t_i)) \right\| \\ &> \frac{\varepsilon}{\sup\|\varphi_n\|} .\end{aligned}$$

This contradicts condition *(iii)* since (A_j) is a decreasing sequence in Σ with $\bigcap_j A_j = \emptyset$. Therefore $\sum y_i$ converges in F . Let $y = \sum y_i$. We claim that $(T(\Phi_n))$ is weakly convergent to y . Let $\varepsilon > 0$ and let $y' \in F'$ with $\|y'\| \leq 1$, then there exist $n_0 \in \mathbb{N}$ and $k \in \mathbb{N}$ such that

$$\left\| \sum_{i > n_0} y_i \right\| < \varepsilon/4 , \quad \hat{m}(\{t_i : i > n_0\}) < \frac{\varepsilon}{4\sup\|\varphi_n\|+1} , \quad \frac{1}{k} < \frac{\varepsilon}{4\|T\|+1} ,$$

and

$$\left|\left\langle \sum_{i=1}^{n_0} m(\{t_i\})(\varphi_n(t_i)) - \sum_{i=1}^{n_0} y_i, y' \right\rangle\right| < \varepsilon/4 \quad \text{for all } n \geq k .$$

If $\hat{T}$ is the extension of T to $B(\Sigma, E)$ and we put $B = \{t_i : i > n_0\}$ then for each $n \geq k$ one has

$$\begin{aligned}|\langle T(\Phi_n)-y,\, y'\rangle| &\le |\langle \hat{T}(\Phi_n-\varphi_n),\, y'\rangle| + |\langle \hat{T}(\varphi_n)-y,\, y'\rangle| \\ &< \frac{\varepsilon}{4} + \left|\left\langle \int_B \varphi_n dm,\, y'\right\rangle\right| + \left|\left\langle \left(\int_{K\setminus B} \varphi_n dm\right)-y,\, y'\right\rangle\right| \\ &\le \frac{\varepsilon}{4} + \|\varphi_n\|\hat{m}(B) + \left|\left\langle \sum_{i>n_0} y_i,\, y'\right\rangle\right| \\ &\quad + \left|\left\langle \sum_{i=1}^{n_0} m(\{t_i\})(\varphi_n(t_i)) - \sum_{i=1}^{n_0} y_i,\, y'\right\rangle\right| < \varepsilon .\end{aligned}$$

Thus we conclude that T is weakly compact.

REMARK. Note that conditions *(i)*, *(ii)* and *(iii)* of Lemma 6 are necessary for an operator $T : C(K, E) \to F$ to be weakly compact but, in general, they are not sufficient.

THEOREM 7. *Suppose that* K *is scattered. Then* $C(K, E)$ *has either the reciprocal Dunford-Pettis property, or the Dieudonné property, or property* V *if and only if* E *has the same property.*

Proof. We only consider the case of the Dieudonné property. The rest can be proved in the same way.

If $C(K, E)$ has the Dieudonné property it is clear that E has it too.

Assume that E has the Dieudonné property.

(A) Let us first suppose that K is metrizable. Let $T : C(K, E) \to F$ be an operator which maps weakly Cauchy sequences into weakly convergent ones. Then T is unconditionally converging and, by Theorem 3 of [3], its representing measure m verifies conditions *(i)* and *(iii)* of Lemma 6. For each $A \in \Sigma$ the map $\tau_A : E \to B(\Sigma, E)$ defined by $\tau_A(x) = x\chi_A$ is a bounded linear map. So it follows from Theorem 2 that the operator $m(A) = \hat{T}\tau_A : E \to F$ transforms weakly Cauchy sequences into weakly convergent sequences. Since E has the Dieudonné property $m(A) : E \to F$ is weakly compact. Therefore, according to Lemma 6, T is weakly compact.

(B) For a general K the same method used in [1], page 236, and the fact that a metrizable quotient space of a scattered space is scattered (see 8.5.3 of [8]), proves that $C(K, E)$ has the Dieudonné property.

The next result is an immediate consequence of 8.5.7, 21.5.1 and 21.5.10 of [8] and Theorem 7, by means of the standard reduction to the case K metrizable.

COROLLARY 8. *$C(K, E)$ has either the reciprocal Dunford-Pettis property, or the Dieudonné property, or property V for every compact K if and only if $C([0, 1], E)$ has the same property.*

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